Curvilinear Co-ordinates

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Upto now we have considered a rectangular Cartesian system, which has the advantage that all the three unit vectors $\hat{i}, \hat{j}, \hat{k}$ are constant unit vectors. In applications, it is often useful to use other Co-ordinate system. We shall discuss now general curvilinear Co-ordinate system

Transformation of Coordinates

Let the rectangular Co-ordinates (x, y, z) of any point be expressed as functions of u, v, w as

$$\left.\begin{aligned} x &= F(u,v,w) \quad \text{or} \quad x = x(u,v,w) \\ y &= G(u,v,w) \quad \text{or} \quad y = y(u,v,w) \\ z &= H(u,v,w) \quad \text{or} \quad z = z(u,v,w) \end{aligned}\right\} \longrightarrow (1)$$

where F, G, H have continuous 1st partial derivatives in a certain region in which their Jacobian is not zero. Since Jacobian of F, G, H is non-zero, Therefore these are not connected by a functional relation and (1) can be solved for u, v, w in terms of x, y, z.

Since the Jacobian is continuous, its sign cannot change in the region and without loss of generality we can suppose that

Jacobian

② ← $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \underline{r}_u \times \underline{r}_v \cdot \underline{r}_w > 0$

because Jacobian were −ve an interchange of v & w (for example) would make it +ve.

Under these hypotheses the expression ① has a unique inverse in some neighbourhood I_0 of any point (x_0, y_0, z_0) and establish a one to one correspondence between points in xyz and uvw co-ordinate system. In this way ① canbe solved uniquely for u, v, w in terms of x, y, z i.e

$$\left.\begin{aligned} u = f(x,y,z) &= u(x,y,z) \\ v = g(x,y,z) &= v(x,y,z) \\ w = h(x,y,z) &= w(x,y,z) \end{aligned}\right\} \longrightarrow ②$$

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and f, g, h also have continuous 1st partial derivatives. In the neighbourhood I_0 the correspondence

$$(x,y,z) \longleftrightarrow (u,v,w)$$

is one-to-one so that a point $P(x,y,z)$ may also be specified by three numbers (u,v,w) given by ②. Co-ordinates (u,v,w) are called curvilinear co-ordinates of P.

The Jacobian J' of the transformation ② is given by

$$J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \neq 0 \longrightarrow ③$$

from ② and ③

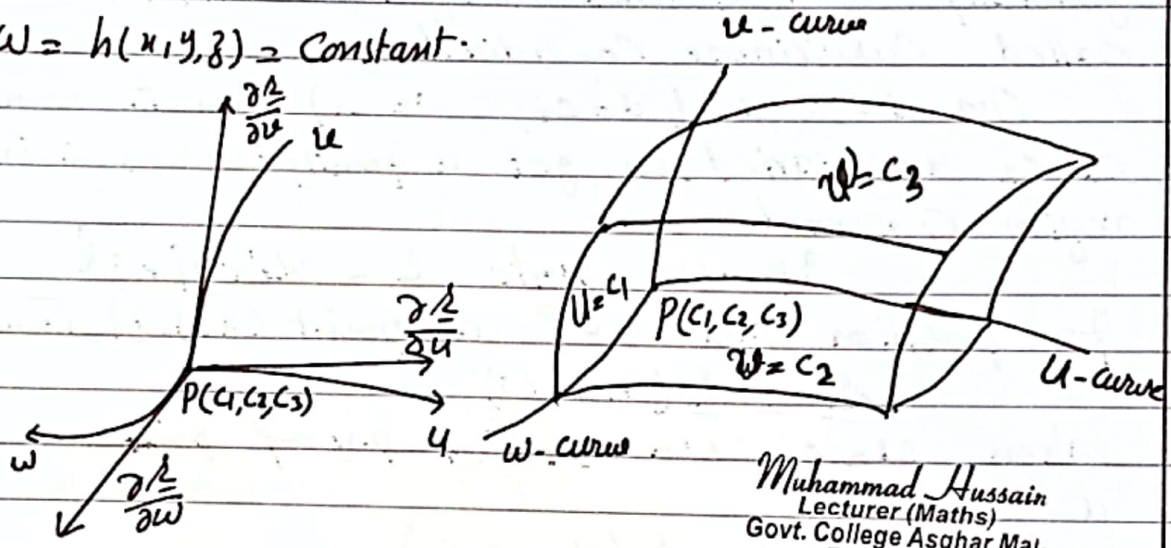
$$JJ' = 1$$

Co-ordinate Surfaces and Co-ordinate Curves

We know that equations in (3) represent three (fan) surfaces when expressed in the form

$U = f(x,y,z) = \text{Constant}$ $v = g(x,y,z) = \text{Constant}$

$w = h(x,y,z) = \text{Constant}$.



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when $P(c_1, c_2, c_3)$ is given, then P is the point of intersection of three surfaces

$U = c_1$ $v = c_2$ $w = c_3$.

These surfaces are called co-ordinates surfaces. The co-ordinate surfaces are generally curved and intersect pairwise in three curves which are called co-ordinate curves.

Thus U-curve is curve along which u varies and v, w remain constant i.e U-curve is intersection of surfaces

$v = c_2$, $w = c_3$

v-curve is the curve along which v varies and u, w remain constant i.e it is intersection of surfaces $U = c_1$ & $w = c_3$.

Similarly w-curve is the intersection of surfaces $U = c_1$, $v = c_2$.

One variable is constant on each of the co-ordinate

Surfaces.

The surface $u = c_1$ on which u remain constant is called u-surface and similarly v-surface and w-surface are defined.

Since three co-ordinate curves are not generally straight lines, therefore co-ordinates are called curvilinear co-ordinates.

On u-curve ($v = c_2$, $w = c_3$) when the numbers c_1, c_3 are varied, we get a family of curves in the region concerned.

If we write $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ for position vector of a point (x, y, z), then

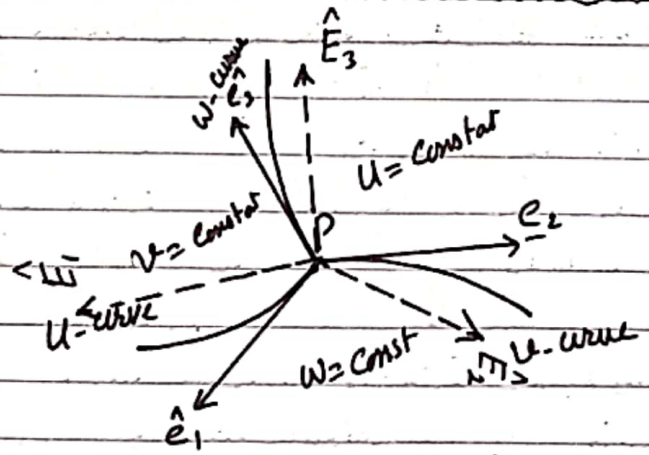
$$\underline{r} = \underline{r}(u, v, w)$$

when $v = c_2$, $w = c_3$, then p.v of point on u-curve is

$$\underline{r} = \underline{r}(u, c_2, c_3)$$

$\frac{\partial \underline{r}}{\partial u}$ is tangent to this curve. Similarly $\frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}$ are tangents to the v-curve and w-curve.

Unit Vectors in Curvilinear Coordinates



Just like base vectors $\hat{i}, \hat{j}, \hat{k}$ (unit orthogonal) of rectangular co-ordinates system, we can introduce suitable set of base vectors for curvilinear co-ordinates. In the most general case, at each point p in space there are two set of base vectors.

First Set The ist set of base vectors has

$$\hat{e}_1 = \hat{e}_u \quad , \hat{e}_2 = \hat{e}_v \quad \hat{e}_3 = \hat{e}_w$$

which are unit tangent vectors at p to the u- v- w- curves respectively are so are defined as

$$\hat{e}_1 = \hat{e}_u = \frac{\frac{\partial \underline{r}}{\partial u}}{\left|\frac{\partial \underline{r}}{\partial u}\right|} = \frac{\frac{\partial \underline{r}}{\partial u}}{h_1}$$

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where $h_1 = \left|\frac{\partial \underline{r}}{\partial u}\right| = h_u$

$$\hat{e}_2 = \hat{e}_v = \frac{\frac{\partial \underline{r}}{\partial v}}{\left|\frac{\partial \underline{r}}{\partial v}\right|} = \frac{\frac{\partial \underline{r}}{\partial v}}{h_2}, \text{ where } h_2 = \left|\frac{\partial \underline{r}}{\partial v}\right| = h_v$$

$$\hat{e}_3 = \hat{e}_w = \frac{\frac{\partial \underline{r}}{\partial w}}{\left|\frac{\partial \underline{r}}{\partial w}\right|} = \frac{\frac{\partial \underline{r}}{\partial w}}{h_3} \quad h_3 = \left|\frac{\partial \underline{r}}{\partial w}\right| = h_w$$

The quantities $h_1 = h_u$, $h_2 = h_v$, $h_3 = h_w$ are called metrical co-efficients and sometimes are called scale factors. $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are in the directions of increasing u, v, w - curves respectively. In general h_1, h_2, h_3 are functions of u, v, w and $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$. Hence $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are also functions of u, v, w.

Second Set# Since ∇u is a vector a P normal to the surface $u =$ constant, therefore a unit vector in this direction is given by

$$\hat{E}_u = \hat{E}_1 = \frac{\nabla u}{|\nabla u|}$$

Similarly unit normal vectors to the surfaces $v =$ const & $w =$ const are defined by

$$\hat{E}_v = \hat{E}_2 = \frac{\nabla v}{|\nabla v|} \qquad \hat{E}_w = \hat{E}_3 = \frac{\nabla w}{|\nabla w|}$$

Vectors $\hat{E}_1, \hat{E}_2, \hat{E}_3$ non-coplanar and can be regarded unit basis vectors at P.

Thus at each point P of curvilinear Co-ordinate system there exist, in general two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the Co-ordinate curves and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ normal to the Co-ordinate surfaces. These two sets of vectors generally vary in direction (orientation) from point to point because because the co-ordinates are curved. However the two sets become identical iff the curvilinear co-ordinate system is orthogonal.

Note# $h_u = h_1 = \left|\frac{\partial \underline{r}}{\partial u}\right|$

$$= \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$$

is the "speed" in terms of u with which u-curve is traced.

$\because$ Along u-curve v, w are constants

$\therefore$ Let $\underline{r} = \underline{r}(u, c_2, c_3)$

If s denotes the arc length along u-curve, then

$$\frac{ds}{du} = \left|\frac{d\underline{r}}{du}\right| = \left|\frac{\partial \underline{r}}{\partial u}\right|$$

$$= h_u$$

$$ds = h_u\, du = h_1\, dy$$

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Similarly along v-curve and w-curve elements of distance or arc length elements are given by

$$ds = h_v\, dv = h_2\, dv$$

$$ds = h_w\, dw = h_3\, dw$$

Problem # (a) Show that the three unit vectors $\hat{e}_u, \hat{e}_v, \hat{e}_w$ tangent a pt P to u-, v-, w- curves of curvilinear co-ordinates system can be expressed as

$$\hat{e}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u} \qquad \hat{e}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}$$

$$\hat{e}_w = \frac{1}{h_w}\frac{\partial \vec{r}}{\partial w} \quad \text{where } h_u = \left|\frac{\partial \underline{r}}{\partial u}\right| \quad h_v = \left|\frac{\partial \underline{r}}{\partial v}\right|$$

$$h_w = \left|\frac{\partial \underline{r}}{\partial w}\right|$$

(b) # Show that the three unit vectors $\hat{E}_u, \hat{E}_v, \hat{E}_w$ normal at pt P to the u-, v-, w- surfaces of curvilinear co-ordinates system can be expressed as

$$\hat{E}_u = \frac{1}{H_u}\nabla u \qquad \hat{E}_v = \frac{1}{H_v}\nabla v \qquad \hat{E}_w = \frac{1}{H_w}\nabla w$$

where $H_u = |\nabla u| \quad H_v = |\nabla v| \quad H_w = |\nabla w|$

Sol (a) # If $\underline{r}$ is the p.v from origin of the rectangular co-ordinates system to the point $P(x, y, z)$, then

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

If (u, v, w) are curvilinear co-ordinates of point P, then $\underline{r}$ can be expressed as a function of u, v, w as

$$\underline{r} = \underline{r}(u, v, w)$$

where $x = x(u, v, w) \quad y = y(u, v, w) \quad z = z(u, v, w)$

When V, w are constants, then

$x = x(u, c_2, c_3) \quad y = y(u, c_2, c_3) \quad z = z(u, c_2, c_3)$

which are parametric equations of u- curve

with parameter u. Hence a tangent vector at P to u-curve is $\frac{\partial \underline{r}}{\partial u}$

Similarly $\frac{\partial \underline{r}}{\partial v}$, $\frac{\partial \underline{r}}{\partial w}$

are tangent vectors at P to v-, w- curves.

In general $\left|\frac{\partial \vec{r}}{\partial u}\right|$, $\left|\frac{\partial \vec{r}}{\partial v}\right|$, $\left|\frac{\partial \vec{r}}{\partial w}\right|$ are not equal to unity.

Hence if $h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|$, $h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|$

$h_w = \left|\frac{\partial \vec{r}}{\partial w}\right|$, then

$$\hat{e}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u} \qquad \hat{e}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}$$

$$\hat{e}_w = \frac{1}{h_w}\frac{\partial \vec{r}}{\partial w}$$

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(b) # Let $\underline{r}$ be p.v from the orgin of a rectangular co-ordinate system to the point $P(x,y,z)$, then

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

If (u, v, w) are curvillinear co-ordinates of point P, then

$$\underline{r} = \underline{r}(u, v, w)$$

If V, W are constant i.e

$$V(x,y,z) = c_2, \text{ \& } W(x,y,z) = c_3$$

Then

$x = x(u, c_2, c_3)$ $y = y(u, c_2, c_3)$ $z = z(u, c_2, c_3)$ are parametric equations of u-curve with parameter

The vector ∇U is perpendicular to the surface $U(x,y,z) = c_1$ at P

Similarly ∇v, ∇w are normals to v- and w- surfaces. Hence if

$H_u = |\nabla u|$ $H_v = |\nabla v|$ $H_w = |\nabla w|$

, then

$\hat{E}_u = \frac{1}{H_u}\nabla u$ $\hat{E}_v = \frac{1}{H_v}\nabla v$

$\hat{E}_w = \frac{1}{H_w}\nabla w$ are three unit vectors normal at P to the Co-ordinates surfaces respectively

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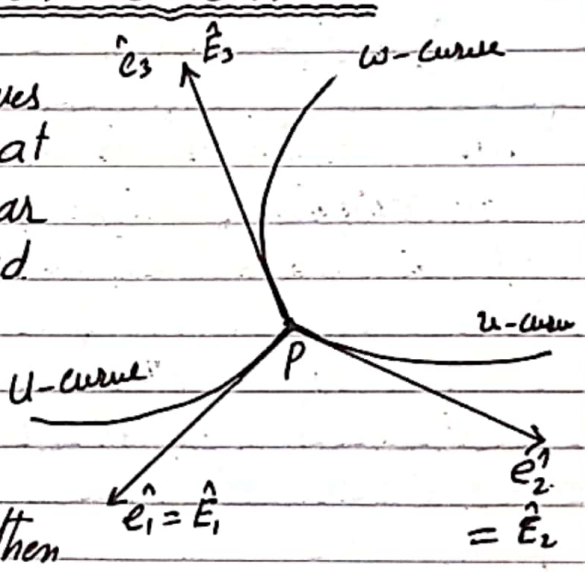
Orthogonal Curvilinear Coordinates



If the Co-ordinate curves intersect at right angles at every point, then curvilinear Co-ordinate system is called orthogonal. In other words if the tangent to Co-ordinate curves through each point are orthogonal to each other, then Corresponding curvilinear Co-ordinates are called curvilinear Co-ordinates.

Clearly if the Co-ordinate curves intersect each other orthogonally, then the Corresponding Co-ordinate surfaces $u = \text{const}$ $v = \text{const}$ $w = \text{const}$ also intersect orthogonally and the two sets of unit vectors $\hat{e}_u, \hat{e}_v, \hat{e}_w$ & $\hat{E}_u, \hat{E}_v, \hat{E}_w$ are the same. In an orthogonal Co-ordinate system, the unit vectors $\hat{e}_u, \hat{e}_v, \hat{e}_w$ are mutually orthogonal at every point. i.e

$$\hat{e}_u \cdot \hat{e}_v = \hat{e}_u \cdot \hat{e}_\omega = \hat{e}_v \cdot \hat{e}_\omega = 0$$

and

$$\hat{e}_u \cdot \hat{e}_u = \hat{e}_v \cdot \hat{e}_v = \hat{e}_\omega \cdot \hat{e}_\omega = 1$$

$\Rightarrow \hat{e}_u, \hat{e}_v, \hat{e}_\omega$ are self resiprocal in case of orthogonal curvilinear co-ordinates.

Further if the system is right handed, then

$$[\hat{e}_u \; \hat{e}_v \; \hat{e}_\omega] = 1$$

and we have

$$\hat{e}_u \times \hat{e}_v = \hat{e}_\omega \qquad \hat{e}_v \times \hat{e}_\omega = \hat{e}_u \qquad \hat{e}_\omega \times \hat{e}_u = \hat{e}_v$$

$$\hat{e}_u \times \hat{e}_\omega = -\hat{e}_v$$

$$\hat{e}_\omega \times \hat{e}_v = -\hat{e}_u$$

$$\hat{e}_v \times \hat{e}_u = -\hat{e}_\omega$$

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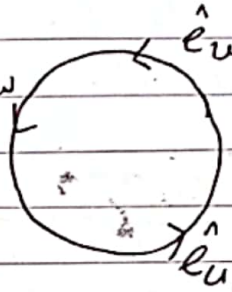

The vector $\vec{A}$ in an orthogonal curvilinear co-ordinates system can be expressed in terms of unit vectors $\hat{e}_u, \hat{e}_v, \hat{e}_\omega$ as

$$\vec{A} = A_1 \hat{e}_u + A_2 \hat{e}_v + A_3 \hat{e}_\omega$$

Where A_1, A_2, A_3 are components of vector $\vec{A}$

Relation Between $\hat{e}_u, \hat{e}_v, \hat{e}_\omega$ & $\hat{E}_u, \hat{E}_v, \hat{E}_\omega$ For Orthogonal Curvilinar Coordinates

Problem# If a system of curvilinear coordinates is orthogonal, then prove that

(a) # $\hat{e}_u = \hat{E}_u \qquad \hat{e}_v = \hat{E}_v \qquad \hat{e}_\omega = \hat{E}_\omega$

(b) # $|\nabla u| = h_u^{-1} \qquad |\nabla v| = h_v^{-1} \qquad h_\omega^{-1} = |\nabla \omega|$

Sol Let us consider

$$\nabla V \cdot \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}\right)$$

$$= \frac{\partial V}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u}.$$

$$\nabla V \cdot \frac{\partial \vec{r}}{\partial u} = \frac{\partial V}{\partial u} \quad \rightarrow (1)$$

But U & V are independent variables. So

$$\frac{\partial V}{\partial u} = 0$$

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(1) becomes

$$\nabla V \cdot \frac{\partial \vec{r}}{\partial u} = 0 \quad \rightarrow (2)$$

Similarly $\nabla W \cdot \frac{\partial \vec{r}}{\partial u} = 0 \quad \rightarrow (3)$

From (2) and (3) we conclude that tangent vector $\frac{\partial \vec{r}}{\partial u}$ is orthogonal to vectors ∇V & ∇W.

Next we show that $\nabla U, \nabla V, \nabla W$ are mutually orthogonal vectors for orthogonal curvilinear co-ordinates.

Since the curvilinear co-ordinates are orthogonal, therefore

$$\hat{e}_u \cdot \hat{e}_v = 0 \quad \rightarrow (4)$$

$$\hat{e}_u \cdot \hat{e}_w = 0 \quad \rightarrow (5)$$

$$\hat{e}_v \cdot \hat{e}_w = 0 \quad \rightarrow (6)$$

Since the co-ordinate surfaces are also orthogonal in such case, therefore

$$\hat{E}_u \cdot \hat{E}_v = 0 \quad \rightarrow (7) \Rightarrow \frac{\nabla U}{|\nabla U|} \cdot \frac{\nabla V}{|\nabla V|} = 0$$

$$\hat{E}_u \cdot \hat{E}_w = 0 \quad \rightarrow (8) \Rightarrow \frac{\nabla U}{|\nabla U|} \cdot \frac{\nabla W}{|\nabla W|} = 0$$

$$\hat{E}_v \cdot \hat{E}_w = 0 \quad \rightarrow (9) \Rightarrow \frac{\nabla V}{|\nabla V|} \cdot \frac{\nabla W}{|\nabla W|} = 0$$

Equations ⑦, ⑧ & ⑨ show that

$\nabla u \cdot \nabla v = 0 \quad \nabla u \cdot \nabla w = 0 \quad \nabla v \cdot \nabla w = 0$

$\Rightarrow \nabla u, \nabla v, \nabla w$ are mutually orthogonal

Now $\frac{\partial \underline{r}}{\partial u}$ which is $\perp$ar to both $\nabla v, \nabla w$ will be parallel to ∇u because $\nabla u, \nabla v, \nabla w$ are mutually orthogonal. Hence $\frac{\partial \underline{r}}{\partial u}$ is parallel to vector ∇u and the unit vectors along both are identical i.e

$$\frac{\frac{\partial \underline{r}}{\partial u}}{\left|\frac{\partial \underline{r}}{\partial u}\right|} = \frac{\nabla u}{|\nabla u|} \Rightarrow \hat{e}_u = \hat{E}_u$$

Similarly we can show that $\hat{e}_v = \hat{E}_v \quad \hat{e}_w = \hat{E}_w$

(b)# Since ∇u is parallel to $\hat{e}_u$, therefore

$\hat{e}_u = \lambda_1 \nabla u \rightarrow ①$ where λ_1 is scalar.

$\Rightarrow \hat{e}_u \cdot \frac{\partial \underline{r}}{\partial u} = \lambda_1 \nabla u \cdot \frac{\partial \underline{r}}{\partial u}$

$\Rightarrow \left|\frac{\partial \underline{r}}{\partial u}\right| = \lambda_1 \nabla u \cdot \frac{\partial \underline{r}}{\partial u}$

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$h_u = \lambda_1 \nabla u \cdot \frac{\partial \underline{r}}{\partial u} \rightarrow ②$

Now from

$\underline{r} = \underline{r}(u, v, w)$

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv + \frac{\partial \underline{r}}{\partial w} dw \rightarrow ③$$

Dot Multiplying by ∇u

$$\nabla u \cdot d\underline{r} = \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial u}\right) du + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial v}\right) dv + \left(\nabla v \cdot \frac{\partial \underline{r}}{\partial w}\right) dw$$

$$du = \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial u}\right) du + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial v}\right) dv + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial w}\right) dw$$

Comparing co-efficients of du, dv, dw $\because d\varphi = \nabla \varphi \cdot d\underline{r}$ we have

$$\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial v} = 0 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial w} = 0 \quad \rightarrow (4)$$

Similarly dot multiplying (3) by ∇v & ∇w, we have

$$\nabla v \cdot \frac{\partial \underline{r}}{\partial u} = 0 \qquad \nabla v \cdot \frac{\partial \underline{r}}{\partial v} = 1 \qquad \nabla v \cdot \frac{\partial \underline{r}}{\partial w} = 0 \quad \rightarrow (5)$$

and

$$\nabla w \cdot \frac{\partial \underline{r}}{\partial u} = 0 \qquad \nabla w \cdot \frac{\partial \underline{r}}{\partial w} = 0 \qquad \nabla w \cdot \frac{\partial \underline{r}}{\partial w} = 1 \quad \rightarrow (6)$$

using $\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1$ in (2)

$$h_u = \lambda_1 \cdot 1 \Rightarrow \lambda_1 = h_u$$

putting in (1)

$$\hat{e}_u = h_u \nabla u \longrightarrow \text{Result}$$

Similarly $\hat{e}_v$ is parallel to ∇v, therefore

$$\hat{e}_v = \lambda_2 \nabla v \quad \text{where } \lambda_2 \text{ is scalar} \quad \rightarrow (7)$$

$$\Rightarrow \hat{e}_v \cdot \frac{\partial \underline{r}}{\partial v} = \lambda_2 \nabla v \cdot \frac{\partial \underline{r}}{\partial v}$$

$$|\hat{e}_v| \left|\frac{\partial \underline{r}}{\partial v}\right| \cos 0^\circ = \lambda_2 \nabla v \cdot \frac{\partial \underline{r}}{\partial v}$$

$$h_v = \lambda_2 \cdot 1$$

$$h_v = \lambda_2$$

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using in (7)

$$\hat{e}_v = h_v \nabla v \longrightarrow \text{Result}$$

Since $\hat{e}_w$ is parallel to ∇w, therefore

$$\hat{e}_w = \lambda_3 \nabla w \quad \rightarrow (8) \quad \text{where } \lambda_3 \text{ is scalar}$$

$$\hat{e}_w \cdot \frac{\partial \underline{r}}{\partial w} = \lambda_3 \nabla w \cdot \frac{\partial \underline{r}}{\partial w}$$

$h_w = \lambda_3 \cdot 1$ from ⑥ $\nabla w \cdot \frac{\partial \underline{r}}{\partial w} = 1$

putting $h_w = \lambda_3$ in ⑧

$$\hat{e}_w = h_w \nabla w \longrightarrow \text{Result}$$

Thus $\frac{1}{h_u} = |\nabla u|$, $\frac{1}{h_v} = |\nabla v|$, $\frac{1}{h_w} = |\nabla w|$

Note# from $\hat{e}_u = h_u \nabla u$ $\hat{e}_v = h_v \nabla v$ $\hat{e}_w = h_w \nabla w$, we can prove that

for orthogonal Co-ordinates

$\hat{e}_u = \hat{E}_u$ $\hat{e}_v = \hat{E}_v$ $\hat{e}_w = \hat{E}_w$

as

$$\hat{E}_u = \frac{\nabla u}{|\nabla u|} = h_u \nabla u \quad \because \frac{1}{|\nabla u|} = h_u$$

$\hat{E}_u = \hat{e}_u$ $(\because h_u \nabla u = \hat{e}_u)$

Similarly $\hat{E}_v = \hat{e}_v$ & $\hat{E}_w = \hat{e}_w$

Problem# In an orthogonal Curvilinear Co-ordinates system show that

(a)# $h_u = \frac{1}{H_u}$ $h_v = \frac{1}{H_v}$ $h_w = \frac{1}{H_w}$

(b)# $\hat{e}_u = h_u \nabla u$ $\hat{e}_v = h_v \nabla v$ $\hat{e}_w = h_w \nabla w$

(c)# $[\nabla u \; \nabla v \; \nabla w] = \frac{1}{h_u h_v h_w}$

Sol# (a)# we have by definition

$$\hat{e}_u = \frac{1}{h_u} \frac{\partial \underline{r}}{\partial u} \quad \text{and} \quad \hat{E}_u = \frac{1}{H_u} \nabla u$$

Therefore $\hat{E}_u \cdot \hat{e}_u = \left(\frac{1}{H_u} \nabla u\right) \cdot \left(\frac{1}{h_u} \frac{\partial \underline{r}}{\partial u}\right)$

$$= \frac{1}{h_u H_u}\left(\frac{\partial \underline{r}}{\partial u} \cdot \nabla u\right) \longrightarrow (1)$$

Now $\underline{r} = \underline{r}(u, v, w)$

$$\Rightarrow d\underline{r} = \frac{\partial \underline{r}}{\partial u}du + \frac{\partial \underline{r}}{\partial v}dv + \frac{\partial \underline{r}}{\partial w}dw \longrightarrow (2)$$

Dot multiplying by ∇u

$$\nabla u \cdot d\underline{r} = \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial u}\right)du + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial v}\right)du + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial w}\right)dw$$

By $d\varphi = \nabla\varphi \cdot d\underline{r}$, we have $\nabla u \cdot d\underline{r} = du$

and

$$du = \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial u}\right)du + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial v}\right)dv + \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial w}\right)dw$$

Comparing co-efficients of du, dv, dw, we have

$$\boxed{\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial v} = 0 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial w} = 0}$$

using $\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1$ in (1)

$$\hat{e}_u \cdot \hat{E}_u = \frac{1}{h_u H_u}(1) = \frac{1}{h_u H_u}$$

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Similarly it can be proved that

$$h_v = \frac{1}{H_v} \quad \& \quad h_w = \frac{1}{H_w}$$

(b) # For orthogonal co-ordinates, we have

$$\hat{e}_u = \hat{E}_u = \frac{1}{H_u}\nabla u = h_u \nabla u$$

Similarly we have

$$\hat{e}_v = h_v \nabla v \qquad \hat{e}_w = h_w \nabla w$$

(C) # $[\nabla u \; \nabla v \; \nabla w] = \nabla u \cdot \nabla v \times \nabla w$

$$= \frac{1}{h_u}\hat{e}_u \cdot \frac{1}{h_v}\hat{e}_v \times \frac{1}{h_w}\hat{e}_w = \frac{1}{h_u h_v h_w}[\hat{e}_u \hat{e}_v \hat{e}_w]$$

$\because$ Curvilinear Coordinates are orthogonal

$\therefore \quad [\hat{e}_u \hat{e}_v \hat{e}_w] = 1$

$\Rightarrow [\nabla u \; \nabla v \; \nabla w] = \frac{1}{h_u h_v h_w}$ (Proved)

Jacobian For Transformation Between Cartesian and Curvilinar Coordinates

Problem # If u, v, w are orthogonal curvilinear Co-ordinates, show that the Jacobian of x, y, z with respect to u, v, w is

$$J\left(\frac{x, y, z}{u, v, w}\right) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = h_u h_v h_w.$$

Also Give physical interpretation for taking this Jacobian not identically zero.

Also prove that

$$J\left(\frac{x, y, z}{u, v, w}\right) = \frac{\partial \underline{r}}{\partial u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}$$

Sol # $J\left(\frac{x, y, z}{u, v, w}\right) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$

$= \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}\right) \cdot \left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j} + \frac{\partial z}{\partial v}\hat{k}\right) \times \left(\frac{\partial x}{\partial w}\hat{i} + \frac{\partial y}{\partial w}\hat{j} + \frac{\partial z}{\partial w}\hat{k}\right)$

$= \frac{\partial \underline{r}}{\partial u} \cdot \frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w} = h_u \hat{e}_u \cdot h_v \hat{e}_v \times h_w \hat{e}_w$

$= h_u h_v h_w \; \hat{e}_u \cdot \hat{e}_v \times \hat{e}_w$

$= h_u h_v h_w \cdot 1 \rightarrow (1)$ $\because$ Co-ordinates are orthogonal $\therefore \hat{e}_u \cdot \hat{e}_v \times \hat{e}_w = 1$

Physical Interpretation # If the Jacobian equals zero identically, then $\frac{\partial \underline{r}}{\partial u}, \frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}$ are co-planar and the curvilinear co-ordinate transformation breaks down i.e there is a relation between x, y, z of the form $F(x,y,z)=0$. We therefore require the Jacobian to be different from zero

Proof of $J\left(\frac{x,y,z}{u,v,w}\right) J\left(\frac{u,v,w}{x,y,z}\right) = 1$

We have $$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\right) \cdot \left(\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right) \times \left(\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right)$$

$$= \underline{\nabla} u \cdot \underline{\nabla} v \times \underline{\nabla} w = [\underline{\nabla} u \; \underline{\nabla} v \; \nabla w]$$

For orthogonal co-ordinates

$$\hat{e}_u = h_u \nabla u \qquad \hat{e}_v = h_v \underline{\nabla} v \qquad \hat{e}_w = h_w \nabla w$$

$$\Rightarrow J\left(\frac{u,v,w}{x,y,z}\right) = \frac{1}{h_u}\hat{e}_u \cdot \frac{1}{h_v}\hat{e}_v \times \frac{1}{h_w}\hat{e}_w$$

$$= \frac{1}{h_u h_v h_w}\, \hat{e}_u \cdot \hat{e}_v \times \hat{e}_w$$

$$= \frac{1}{h_u h_v h_w}$$

Thus

$$J\left(\frac{x,y,z}{u,v,w}\right) J\left(\frac{u,v,w}{x,y,z}\right) = h_u h_v h_w \cdot \frac{1}{h_u h_v h_w} = 1$$

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Reciprocal Systems of Vecctors $\frac{\partial \underline{r}}{\partial u}, \frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}$ and $\nabla u, \nabla v, \nabla w$

Problem# If u, v, w are general Coordinates, then shaw that two systems of vectors $\frac{\partial \underline{r}}{\partial u}, \frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}$ and $\nabla u, \nabla v, \nabla w$ are reciprocal systems.

Sol# If u, v, w are curvilinear Co-ordinates of a point P whose P.V w.r.t origin of rectangular Co-ordinate system is $\underline{r}$, then we have

$$\underline{r} = \underline{r}(u, v, w)$$

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We are to Show that

$$\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1 \qquad \nabla v \cdot \frac{\partial \underline{r}}{\partial v} = 1 \qquad \nabla w \cdot \frac{\partial \underline{r}}{\partial w} = 1$$

$$\nabla u \cdot \frac{\partial \underline{r}}{\partial v} = 0 = \nabla u \cdot \frac{\partial \underline{r}}{\partial w} = \nabla v \cdot \frac{\partial \underline{r}}{\partial u} = \nabla v \cdot \frac{\partial \underline{r}}{\partial w}$$

$$\nabla w \cdot \frac{\partial \underline{r}}{\partial u} = 0 = \nabla w \cdot \frac{\partial \underline{r}}{\partial v}.$$

Now $$d\underline{r} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv + \frac{\partial \underline{r}}{\partial w} dw \longrightarrow (1)$$

Dot Multiplying (1) by ∇u

$$\nabla u \cdot d\underline{r} = \left(\nabla u \cdot \frac{\partial \underline{r}}{\partial u}\right) du + \nabla u \cdot \frac{\partial \underline{r}}{\partial v} du + \nabla u \cdot \frac{\partial \underline{r}}{\partial w} dw$$

$\because \nabla \varphi \cdot d\underline{r} = d\varphi$

$$\Rightarrow du = \nabla u \cdot \frac{\partial \underline{r}}{\partial u} du + \nabla u \cdot \frac{\partial \underline{r}}{\partial v} dv + \nabla u \cdot \frac{\partial \underline{r}}{\partial w} dw$$

Comparing Co-efficients of du, dv, dw

$$\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 1 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial v} = 0 \qquad \nabla u \cdot \frac{\partial \underline{r}}{\partial w} = 0$$

Similarly Do Mulliplying (1) by ∇v, ∇w, we can prove

$\nabla u \cdot \frac{\partial \underline{r}}{\partial u} = 0 \quad \nabla u \cdot \frac{\partial \underline{r}}{\partial v} = 1 \quad \nabla u \cdot \frac{\partial \underline{r}}{\partial w} = 0$

and

$\nabla w \cdot \frac{\partial \underline{r}}{\partial u} = 0 \quad \nabla w \cdot \frac{\partial \underline{r}}{\partial v} = 0 \quad \nabla w \cdot \frac{\partial \underline{r}}{\partial w} = 1$

Thus the two systems are reciprocal to each other.

Alternate Method

When the p.v $\underline{r}$ is regarded as function of x, y, z, we have

$$\nabla \underline{r} = \frac{\partial \underline{r}}{\partial x}\hat{i} + \frac{\partial \underline{r}}{\partial y}\hat{j} + \frac{\partial \underline{r}}{\partial z}\hat{k}$$

$$= \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k} = \underline{I} \longrightarrow (1)$$

$\frac{\partial \underline{r}}{\partial x} = \hat{i}$

$\frac{\partial \underline{r}}{\partial y} = \hat{j}$

$\frac{\partial \underline{r}}{\partial z} = \hat{k}$

Now let $\underline{a} = \nabla u \quad \underline{b} = \nabla v, \ \underline{c} = \nabla w$ and $\underline{a}', \underline{b}', \underline{c}'$ be the reciproal to $\underline{a}, \underline{b}, \underline{c}$. when $\underline{r}$ is regarded as a function of u, v, w, then

$$\nabla \underline{r} = \hat{i}\,\underline{r}_x + \hat{j}\,\underline{r}_y + \hat{k}\,\underline{r}_z$$

Now $\underline{r}_x = \frac{\partial \underline{r}}{\partial x} = \frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial x}$

$\underline{r}_y = \frac{\partial \underline{r}}{\partial y} = \frac{\partial \underline{r}}{\partial v} \cdot \frac{\partial v}{\partial y}$

$\underline{r}_z = \frac{\partial \underline{r}}{\partial z} = \frac{\partial \underline{r}}{\partial w} \cdot \frac{\partial w}{\partial z}$

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$$\Rightarrow \nabla \underline{r} = \hat{i}\frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial \underline{r}}{\partial v}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial \underline{r}}{\partial w} \cdot \frac{\partial w}{\partial z}$$

$$= \frac{\partial \underline{r}}{\partial u}\left(\frac{\partial u}{\partial x}\hat{i}\right) + \frac{\partial \underline{r}}{\partial v}\left(\frac{\partial v}{\partial y}\hat{j}\right) + \frac{\partial \underline{r}}{\partial w}\left(\frac{\partial w}{\partial z}\hat{k}\right)$$

$$= \frac{\partial \underline{r}}{\partial u}\nabla u + \frac{\partial \underline{r}}{\partial v}\nabla v + \frac{\partial \underline{r}}{\partial w}\nabla w$$

$$= \underline{a}\frac{\partial \underline{r}}{\partial u} + \underline{b}\frac{\partial \underline{r}}{\partial v} + \underline{c}\frac{\partial \underline{r}}{\partial w} \longrightarrow (2)$$

By (1) & (2)

$$\underline{I} = \underline{a}\frac{\partial \underline{r}}{\partial u} + \underline{b}\frac{\partial \underline{r}}{\partial v} + \underline{c}\frac{\partial \underline{r}}{\partial w} \longrightarrow (3)$$

on dot multiplying (3) by $\underline{a}'$

$$\underline{a}' \cdot \underline{I} = \underline{a}' \cdot \underline{a}\frac{\partial \underline{r}}{\partial u} + \underline{a}' \cdot \underline{b}\frac{\partial \underline{r}}{\partial v} + \underline{a}' \cdot \underline{c}\frac{\partial \underline{r}}{\partial w}$$

$\because$ $\underline{a}, \underline{b}, \underline{c}$ & $\underline{a}', \underline{b}', \underline{c}'$ are reciprocal

$\therefore$ $\underline{a} \cdot \underline{a}' = 1 \quad \underline{b} \cdot \underline{b}' = 1 = \underline{c} \cdot \underline{c}'$

and $\underline{a}' \cdot \underline{b} = 0 = \underline{a}' \cdot \underline{c}$

$\underline{b}' \cdot \underline{a} = 0 = \underline{b}' \cdot \underline{c}$

$\underline{c}' \cdot \underline{a} = 0 = \underline{c}' \cdot \underline{b}$

$$[\underline{a}'\underline{b}'\underline{c}'] = \frac{1}{[\underline{a}\underline{b}\underline{c}]}$$

$$\underline{a}' = \frac{\underline{b} \times \underline{c}}{[\underline{a}\underline{b}\underline{c}]}$$

$$\underline{b}' = \frac{\underline{c} \times \underline{a}}{[\underline{a}\underline{b}\underline{c}]}$$

$$\underline{a}' \cdot \underline{I} = \underline{a}' \cdot (i\hat{i} + j\hat{j} + k\hat{k})$$

$$= (\underline{a}' \cdot \hat{i})\hat{i} + (\underline{a}' \cdot \hat{j})\hat{j} + (\underline{a}' \cdot \hat{k})\hat{k}$$

$$= a'_x \hat{i} + a'_y \hat{j} + a'_z \hat{k} = \underline{a}'$$

$$\Rightarrow \quad \underline{a}' = \frac{\partial \underline{r}}{\partial u}.$$

Similarly on dot multiplying (3) by $\underline{b}', \underline{c}'$ as prefactors, we get

$$\underline{b}' = \frac{\partial \underline{r}}{\partial v} \qquad \underline{c}' = \frac{\partial \underline{r}}{\partial w}$$

Thus

$$\underline{a}' = \frac{\partial \underline{r}}{\partial u} \qquad \underline{b}' = \frac{\partial \underline{r}}{\partial v} \qquad \underline{c}' = \frac{\partial \underline{r}}{\partial w}$$

are reciprocal to

$$\underline{a} = \nabla u \qquad \underline{b} = \nabla v \qquad \underline{c} = \nabla w$$

i.e

$$[\nabla u \; \nabla v \; \nabla w]\left[\frac{\partial \underline{r}}{\partial u} \; \frac{\partial \underline{r}}{\partial v} \; \frac{\partial \underline{r}}{\partial w}\right] = 1$$

Note # From the properties of above reciprocal sets, we have.

$$\nabla U = \frac{\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}}{\left[\frac{\partial \underline{r}}{\partial u}\ \frac{\partial \underline{r}}{\partial v}\ \frac{\partial \underline{r}}{\partial w}\right]} = \frac{\underline{r}_v \times \underline{r}_w}{J}$$

$$\nabla V = \frac{\underline{r}_w \times \underline{r}_u}{J}$$

$$\nabla W = \frac{\underline{r}_u \times \underline{r}_v}{J}$$

$$[\nabla U\ \nabla V\ \nabla W] = \frac{\partial(U,V,W)}{\partial(x,y,z)} = \frac{1}{J}$$

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Transformation of arc Length, Area & Volume Elements under Orthogonal Curvilinear Coordinates

OR

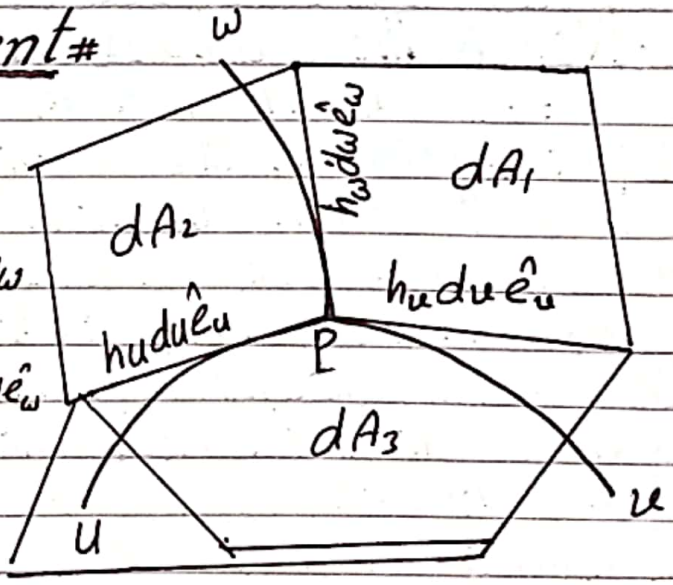
Expressions for Arc Length, Area & Volume Elements in Orthogonal Curvilinear Coordinates

(i) Arc Length Element

From $\underline{r} = \underline{r}(u,v,w)$

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u}du + \frac{\partial \underline{r}}{\partial v}dv + \frac{\partial \underline{r}}{\partial w}dw$$

$$= h_u du\,\hat{e}_u + h_v dv\,\hat{e}_v + h_w dw\,\hat{e}_w \quad \rightarrow (1)$$



Then the differential of arc length is determined from

$$(ds)^2 = d\vec{r} \cdot d\vec{r}$$

$$= (h_u du\,\hat{e}_u + h_v dv\,\hat{e}_v + h_w dw\,\hat{e}_w) \cdot (h_u du\,\hat{e}_u + h_v dv\,\hat{e}_v + h_w dw\,\hat{e}_w)$$

For an orthogonal curvilinear Co-ordinates system

$$\hat{e}_u \cdot \hat{e}_u = 1 \quad \hat{e}_u \cdot \hat{e}_v = 0 \quad \hat{e}_u \cdot \hat{e}_w = 0$$
$$\hat{e}_v \cdot \hat{e}_v = 1 \quad \hat{e}_w \cdot \hat{e}_w = 1 \quad \hat{e}_v \cdot \hat{e}_w = 0$$

Hence

$$(ds)^2 = h_u^2 (du)^2 + h_v^2 (dv)^2 + h_w^2 (dw)^2$$

$$= (h_u\, du)^2 + (h_v\, dv)^2 + (h_w\, dw)^2 \rightarrow (2)$$

where ds is differential of arc length of a curve traced and du, dv, dw differential changes along $u-$ $v-$ $w-$ curvilinear curves. Now along $u-$ curve, v, w are constants so that $dv = 0 = dw$. Then the differential arc length along ds_u or ds_1 along u at p is given by (2)

$$ds_u = h_u\, du$$

Similarly the differential arc lengths along $v-$ and $w-$ curves at p are

$$ds_v = ds_2 = h_v\, dv \qquad ds_w = ds_3 = h_w\, dw$$

Note# The numbers h_u, h_v, h_w which in general depend on u, v, w are called scale factors. Now all the co-ordinates u, v, w may not have the same dimensions of length; in fact some of the co-ordinates may be angles as in case of spherical and cylinderical polar coordinates. However all of the quantities ds_u, ds_v, ds_w have dimensions of length. By multiplying du, dv, dw with h_u, h_v, h_w respectively we obtain quantities that have the dimensions of length. Hence the name scalar factors.

$$h_u^2 = \left|\frac{\partial \vec{r}}{\partial u}\right| = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}$$

$$h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$$

In Numerical Problems an easy method to calculate the scale factors, h_u, h_v, h_w is to start with

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \longrightarrow ③$$

Now we substitute for dx, dy, dz

Since

$$x = x(u, v, w) \qquad y = y(u, v, w)$$
$$z = z(u, v, w)$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$$

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After putting dx, dy, dz in ③, we compare the co-efficients in ③ of $(du)^2, (dv)^2, (dw)^2$ with those of $(du)^2, (dv)^2, (dw)^2$ in equation

$$(ds)^2 = (h_u\,du)^2 + (h_v\,dv)^2 + (h_w\,dw)^2.$$

2: Area Element#

From fig area elements are given by

$$dA_1 = |(h_v\,dv\,\hat{e}_v) \times (h_w\,dw\,\hat{e}_w)|$$

$$= h_v h_w |\hat{e}_u \times \hat{e}_w|\,dv\,dw$$

$$= h_v h_w |\hat{e}_u|\,dv\,dw \qquad [\because \hat{e}_u \times \hat{e}_w = \hat{e}_u]$$

$$= h_v h_w \cdot 1 \cdot dv\,dw = h_v h_w\,dv\,dw$$

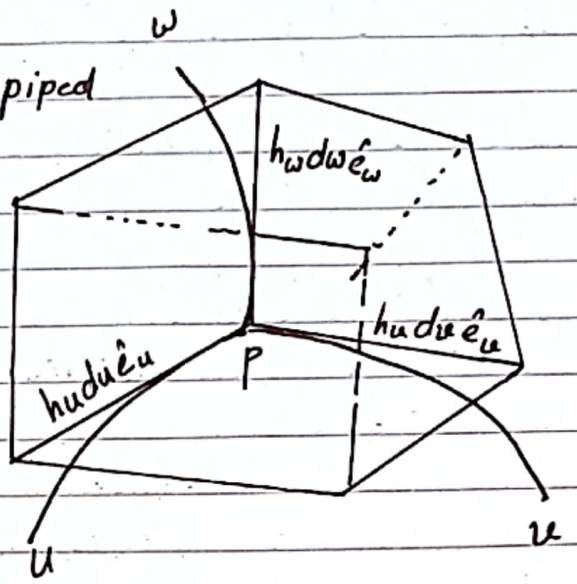
Similarly

$$dA_2 = |(h_u\,du\,\hat{e}_u) \times (h_w\,dw\,\hat{e}_w)| = h_u h_w\,du\,dw$$

$$dA_3 = h_u h_v\,du\,dv$$

3):- Volume Element

Consider the parallelopiped formed by tangents vectors at point P. We know that the absolute value of scalar triple product gives the volume of the parallelopiped.. Thus volume of this small volume element is



$$dV = |(h_u du \hat{e}_u) \cdot (h_v dv \hat{e}_v) \times (h_w dw \hat{e}_w)|$$

$$= h_u h_v h_w \, du\, dv\, dw \, |\hat{e}_u \cdot \hat{e}_v \times \hat{e}_w|$$

$$= h_u h_v h_w \, du\, dv\, dw \cdot 1 \qquad \because |\hat{e}_u \cdot \hat{e}_v \times \hat{e}_w| = 1$$

But we have proved that

$$h_u h_v h_w = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

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$$dV = \frac{\partial(x,y,z)}{\partial(u,v,w)} du\, dv\, dw$$

which is the required formula for the volume element in curvilinear Co-ordinates.

Note # It is clear that when Jacobian vanishes there is no parallelopiped which explains geometrically the significance of the vanishing of a Jacobian which we have already discussed.

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روز محشر عذر ہائے من پذیر
گر تو می بینی حسابم ناگزیر
از نگاہِ مصطفیٰ پنہاں بگیر

اقبال

#Gradient, Divergence, Curl and Laplacian in Orthogonal Curvilinear Coordinates#

(OR Differential Operations in Orthogonal Curvilinear Co-ordinates)

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1)# Expression For Gradient OR Transformation of Gradient in Orthogonal Curvilinear Coordinates

Let $\phi(u,v,w)$ be any scalar point function given in terms of orthogonal curvilinear coordinates u,v,w. Then

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \longrightarrow (1)$$

$\because \quad U = U(x,y,z) \quad V = V(x,y,z), \; W = W(x,y,z)$

$$\phi = \phi\big(U(x,y,z),\, v(x,y,z),\, w(x,y,z)\big)$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x} \longrightarrow (2)$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial y} \longrightarrow (3)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial z} \longrightarrow (4)$$

using (2), (3) and (4) in (1)

$$\nabla\phi = \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x}\right)\hat{i} + \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial y}\right)\hat{j} + \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial\phi}{\partial w}\frac{\partial\phi}{\partial z}\right)\hat{k}$$

$$= \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial v}{\partial z}\hat{k}\right) + \frac{\partial\phi}{\partial w}\left(\frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right)$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial u}\nabla u + \frac{\partial\varphi}{\partial v}\nabla v + \frac{\partial\varphi}{\partial w}\nabla w \longrightarrow (5)$$

This is expression of gradient of φ in general curvilinear Co-ordinates.

But for orthogonal curvilinear Co-ordinates we have the following relations.

$$\nabla u = \frac{\hat{e}_u}{h_u} \qquad \nabla v = \frac{\hat{e}_v}{h_v} \qquad \nabla w = \frac{\hat{e}_w}{h_w}$$

Using these relations in (5)

$$\nabla\varphi = \frac{1}{h_u}\frac{\partial\varphi}{\partial x}\hat{e}_u + \frac{1}{h_v}\frac{\partial\varphi}{\partial y}\hat{e}_v + \frac{1}{h_w}\frac{\partial\varphi}{\partial w}\hat{e}_w$$

Result

which is the required expression for gradien of φ in orthogonal curvilinear Coordinates.

This indicates the operator equivalence

$$\nabla = \frac{\hat{e}_u}{h_u}\frac{\partial}{\partial u} + \frac{\hat{e}_v}{h_v}\frac{\partial}{\partial v} + \frac{\hat{e}_w}{h_w}\frac{\partial}{\partial w}$$

which is the expression for del operator in orthogonal curvilinear Coordinates

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Secod Method

Let $\nabla\varphi = f_1\hat{e}_u + f_2\hat{e}_v + f_3\hat{e}_w \longrightarrow (6)$

where f_1, f_2, f_3 are to be determined.

Since

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u}du + \frac{\partial\vec{r}}{\partial v}dv + \frac{\partial\vec{r}}{\partial w}dw$$

$$= h_u du\hat{e}_u + h_v dv\hat{e}_v + h_w dw\hat{e}_w$$

Dot multiplying by $\nabla\varphi$

$$\nabla\varphi \cdot d\vec{r} = h_u f_1 du + h_v f_2 dv + h_w f_3 dw$$

$$d\varphi = h_u f_1 du + h_v f_2 dv + h_w f_3 dw \longrightarrow (7)$$

But

$$d\varphi = \frac{\partial \varphi}{\partial u} du + \frac{\partial \varphi}{\partial v} dv + \frac{\partial \varphi}{\partial w} dw \longrightarrow (8)$$

Comparing co-efficients of du, dv, dw in (7) and (8), we have

$$f_1 = \frac{1}{h_u}\frac{\partial \varphi}{\partial u} \qquad f_2 = \frac{1}{h_v}\frac{\partial \varphi}{\partial v} \qquad f_3 = \frac{1}{h_w}\frac{\partial \varphi}{\partial w}$$

using in Equation (6), we get

$$\nabla \varphi = \frac{1}{h_u}\frac{\partial \varphi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial \varphi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial \varphi}{\partial w}\hat{e}_w \longrightarrow (9)$$

2) # Expression For Divergence in Orthogonal Curvilinear Coordinates

Let $\underline{A}(u, v, w)$ be a vector point function given in terms of orthogonal curvilinear coordinates u, v, w and let A_u, A_v, A_w be components of $\underline{A}$ along $\hat{e}_u, \hat{e}_v, \hat{e}_w$. Then

$$\underline{A} = A_u \hat{e}_u + A_v \hat{e}_v + A_w \hat{e}_w \longrightarrow (1)$$

We know $\hat{e}_u, \hat{e}_v, \hat{e}_w$ form right handed system. Therefore

$$\hat{e}_u = \hat{e}_v \times \hat{e}_w = h_v h_w \nabla u \times \nabla w \longrightarrow (2)$$

$$\hat{e}_v = \hat{e}_w \times \hat{e}_u = h_w h_u \nabla w \times \nabla u \longrightarrow (3)$$

$$\hat{e}_w = \hat{e}_u \times \hat{e}_v = h_u h_w \nabla u \times \nabla v \longrightarrow (4)$$

Putting these values in (1)

$$\underline{A} = h_v h_w A_u \nabla u \times \nabla w + h_w h_u A_v \nabla w \times \nabla u + h_u h_w A_w \nabla u \times \nabla v \longrightarrow (5)$$

Now

$$\nabla\cdot\underline{A} = \nabla\cdot\left(h_v h_w A_u \nabla V\times\nabla W\right) + \nabla\cdot\left(h_w h_u A_v \nabla W\times\nabla U\right)$$

$$+\nabla\cdot\left(h_u h_v A_w \nabla U\times\nabla V\right) \longrightarrow (6)$$

Now using

$\nabla\cdot(\varphi\underline{F}) = \varphi\nabla\cdot\underline{F} + \underline{F}\cdot\nabla\varphi$

and $\nabla\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot\nabla\times\underline{A} - \underline{A}\cdot\nabla\times\underline{B}$

we have

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$$\nabla\cdot\left(h_v h_w A_u \nabla V\times\nabla W\right)$$

$$= h_v h_w A_u \nabla\cdot(\nabla V\times\nabla W) + (\nabla V\times\nabla W)\cdot\nabla(h_v h_w A_u)$$

$$= h_v h_w A_u\left[\nabla W\cdot(\nabla\times\nabla V) - \nabla V\cdot(\nabla\times\nabla W)\right]$$

$$+(\nabla V\times\nabla W)\cdot\nabla(h_v h_w A_u)$$

$$= 0 + (\nabla V\times\nabla W)\cdot\nabla(h_v h_w A_u)$$

$\because \nabla\times\nabla V = 0 \qquad \nabla\times\nabla W = 0$

$$= (\nabla V\times\nabla W)\cdot\nabla(h_v h_w A_u)$$

$$= \frac{1}{h_v h_w}\left[\hat{e}_u\cdot\nabla(h_v h_w A_u)\right] \longrightarrow (7)$$

by (2) $\nabla V\times\nabla W = \frac{1}{h_v h_w}\hat{e}_u$

We know that in orthogonal curvilinear Co-ordinates.

$$\nabla\varphi = \frac{1}{h_u}\frac{\partial\varphi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial\varphi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial\varphi}{\partial w}\hat{e}_w$$

using this

$$\nabla(h_v h_w A_u) = \frac{1}{h_u}\frac{\partial}{\partial u}(h_v h_w A_u)\hat{e}_u + \frac{1}{h_v}\frac{\partial}{\partial v}(h_v h_w A_u)\hat{e}_v$$

$$+\frac{1}{h_w}\frac{\partial}{\partial w}(h_v h_w A_u)\hat{e}_w$$

$$\hat{e}_u \cdot \nabla (h_v h_w A_u) = \frac{1}{h_u} \frac{\partial}{\partial u}(h_v h_w A_u) \longrightarrow (8)$$

$$[\because \hat{e}_u \cdot \hat{e}_v = 0 = \hat{e}_u \cdot \hat{e}_w]$$

using This value in (7)

$$\nabla \cdot (h_v h_w A_u \nabla v \times \nabla w) = \frac{1}{h_u h_v h_w} \frac{\partial}{\partial u}(h_v h_w A_u) \longrightarrow (9)$$

Similarly

$$\nabla \cdot (h_w h_u^{A_v} \nabla w \times \nabla u) = \frac{1}{h_u h_v h_w} \frac{\partial}{\partial v}(h_u h_w A_v) \longrightarrow (10)$$

$$\nabla \cdot (h_u h_v A_w \nabla u \times \nabla v) = \frac{1}{h_u h_v h_w} \frac{\partial}{\partial w}(h_u h_v A_w) \longrightarrow (11)$$

using (9), (10) & (11) in (6)

$$\nabla \cdot \underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_w h_u A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right]$$

which is the required expression of divergence of a vector in orthogonal curvilinear coordinates.

3)# Expression For Curl in Orthogonal Curvilinear Coordinates

Let $\underline{A}(u,v,w)$ be a vector function in terms of orthogonal co-ordinates u,v,w and let A_u, A_v, A_w be the components of $\underline{A}$ along $\hat{e}_u, \hat{e}_v, \hat{e}_w$. Then

$$\underline{A} = A_u \hat{e}_u + A_v \hat{e}_v + A_w \hat{e}_w$$

But $\hat{e}_u = h_u \nabla u \quad \hat{e}_v = h_v \nabla v \quad h_w \nabla w = \hat{e}_w$ in orthogonal curvilinear coordinates.

so

$$\underline{A} = h_u A_u \nabla u + h_v A_v \nabla v + h_w A_w \nabla w \longrightarrow (1)$$

$$\nabla \times A = \nabla \times [h_u A_u \nabla u] + \nabla \times [h_v A_v \nabla v] + \nabla \times [h_w A_w \nabla w] \longrightarrow (2)$$

Now using

$$\nabla \times (\phi F) = \phi \nabla \times F + \nabla \phi \times F$$

we have

$$\nabla \times [h_u A_u \nabla u] = h_u A_u \nabla \times \nabla u + \nabla (h_u A_u) \times \nabla u$$

$$= 0 + \nabla (h_u A_u) \times \nabla u \quad [\because \nabla \times \nabla u = 0]$$

$$\doteq \nabla (h_u A_u) \times \nabla u \longrightarrow (2)$$

$$= \nabla (h_u A_u) \times \frac{\hat{e}_u}{h_u} \quad \left[\because \nabla u = \frac{\hat{e}_u}{h_u}\right] \longrightarrow (3)$$

$$\because \nabla \phi = \frac{1}{h_u}\frac{\partial \phi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial \phi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial \phi}{\partial w}\hat{e}_w$$

$$\therefore \nabla (h_u A_u) = \frac{1}{h_u}\frac{\partial}{\partial u}(h_u A_u)\hat{e}_u + \frac{1}{h_v}\frac{\partial}{\partial v}(h_u A_u)\hat{e}_v + \frac{1}{h_w}\frac{\partial}{\partial w}(h_u A_u)\hat{e}_w$$

$$\Rightarrow \nabla (h_u A_u) \times \frac{\hat{e}_u}{h_u} = \frac{1}{h_u h_v}\frac{\partial}{\partial u}(h_u A_u)\hat{e}_u \times \hat{e}_u + \frac{1}{h_u h_w}\frac{\partial}{\partial w}(h_u A_u)\hat{e}_w \times \hat{e}_u$$

$$= -\frac{1}{h_u h_v}\frac{\partial}{\partial v}(h_u A_u)\hat{e}_w + \frac{1}{h_u h_w}\frac{\partial}{\partial w}(h_u A_u)\hat{e}_v \longrightarrow (4)$$

$\Rightarrow$ (Similarly)

$$\nabla \times [h_u A_u \nabla u] = \frac{1}{h_u h_w}\frac{\partial}{\partial w}(h_u A_u)\hat{e}_v - \frac{1}{h_u h_v}\frac{\partial}{\partial v}(h_u A_u)\hat{e}_w \longrightarrow (4)$$

Similarly

$$\nabla \times [h_v A_v \nabla u] = \frac{1}{h_v h_u}\frac{\partial}{\partial u}(h_v A_v)\hat{e}_w - \frac{1}{h_v h_w}\frac{\partial}{\partial w}(h_v A_v)\hat{e}_u \longrightarrow (5)$$

$$\nabla \times [h_w A_w \nabla w] = \frac{1}{h_w h_v}\frac{\partial}{\partial v}(h_w A_w)\hat{e}_u - \frac{1}{h_w h_u}\frac{\partial}{\partial u}(h_w A_w)\hat{e}_v \longrightarrow (6)$$

Using ④, ⑤ & ⑥ in ②

$$\nabla \times \underline{A} = \frac{1}{h_u h_w}\frac{\partial}{\partial w}(h_u A_u)\hat{e}_u - \frac{1}{h_u h_v}\frac{\partial}{\partial v}(h_u A_u)\hat{e}_w$$

$$+\frac{1}{h_v h_u}\frac{\partial}{\partial u}(h_v A_v)\hat{e}_w - \frac{1}{h_v h_w}\frac{\partial}{\partial w}(h_v A_v)\hat{e}_u$$

$$+\frac{1}{h_w h_v}\frac{\partial}{\partial v}(h_w A_w)\hat{e}_u - \frac{1}{h_w h_u}\frac{\partial}{\partial u}(h_w A_w)\hat{e}_v$$

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$$= \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w A_w) - \frac{\partial}{\partial w}(h_v A_v)\right]\hat{e}_u$$

$$+\frac{1}{h_u h_w}\left[\frac{\partial}{\partial w}(h_u A_u) - \frac{\partial}{\partial u}(h_w A_w)\right]\hat{e}_v$$

$$+\frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v A_v) - \frac{\partial}{\partial v}(h_u A_u)\right]\hat{e}_w$$

$$= \frac{1}{h_u h_v h_w}\left[h_u\left\{\frac{\partial}{\partial v}(h_w A_w) - \frac{\partial}{\partial w}(h_v A_v)\right\}\right]\hat{e}_u$$

$$+\frac{1}{h_u h_v h_w}\left[h_v\left\{\frac{\partial}{\partial w}(h_u A_u) - \frac{\partial}{\partial u}(h_w A_w)\right\}\right]\hat{e}_v$$

$$+\frac{1}{h_u h_v h_w}\left[h_w\left\{\frac{\partial}{\partial u}(h_v A_v) - \frac{\partial}{\partial v}(h_u A_u)\right\}\right]\hat{e}_w$$

$$= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u A_u & h_v A_v & h_w A_w \end{vmatrix} \longrightarrow (7)$$

Since $\underline{A} = A_u \hat{e}_u + A_v \hat{e}_v + A_w \hat{e}_w$

Therefore $\underline{A}\cdot\hat{e}_u = A_u \quad \underline{A}\cdot\hat{e}_v = A_v \quad \underline{A}\cdot\hat{e}_w = A_w$

using these in ⑦

$$\nabla \times \underline{A} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u \underline{A}\cdot\hat{e}_u & h_v \underline{A}\cdot\hat{e}_v & h_w \underline{A}\cdot\hat{e}_w \end{vmatrix}$$

4:) # Expression For Laplacian in Orthogonal Curvilinear Coordinates

Let $\phi(u, v, w)$ be a scalar function in orthogonal curvilinear coordinates u, v, w.

Then we know that

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$$\nabla\phi = \frac{1}{h_u}\frac{\partial\phi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial\phi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial\phi}{\partial w}\hat{e}_w$$

Now using

$$\nabla\cdot\underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_u h_w A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right]$$

we have

$$\nabla^2\phi = \nabla\cdot\nabla\phi$$

$$= \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w \cdot \frac{1}{h_u}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\phi}{\partial w}\right)\right]$$

Hence Laplacian operator in Curvilinear Co-ordinates is

$$\nabla^2 = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial}{\partial w}\right)\right]$$

The Result.

Problem# Consider the curvilinear Co-ordinates defined for $z \geq 0$ by

$$x = u - v \qquad y = u + v \qquad z = w^2$$

(i) # Find the unit vectors $\hat{e}_u, \hat{e}_v, \hat{e}_w$ and show that they system is orthogonal and right-handed. Also find the scale factors h_u, h_v, h_w

(ii) # Find the expression for $(ds)^2$ and dV

(iii) # Find $\nabla\psi$ in the system for $\psi(u,v,w) = uv + w^2$

(iv) # Find $\nabla \cdot \underline{A}$ and $\nabla \times \underline{A}$ for the vector field

$$\vec{A} = wu\hat{e}_u + wv\hat{e}_v + uv\hat{e}_w$$

(v) # Find $\nabla^2\psi$ if $\psi = u^2 + v^2 + w^2$

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Sol# (i) We need not find the inverse transformation but, it is easy to derive

$$u = \frac{x+y}{2} \qquad v = \frac{x-y}{2} \qquad w = \sqrt{z}$$ (take the +ve square root for definiteness)

The position vector $\underline{r}$ for the system is

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (u-v)\hat{i} + (u+v)\hat{j} + w^2\hat{k}$$

$$\frac{\partial \underline{r}}{\partial u} = \hat{i} + \hat{j} \qquad \frac{\partial \underline{r}}{\partial v} = -\hat{i} + \hat{j} \qquad \frac{\partial \underline{r}}{\partial w} = 2w\hat{k}$$

$$\hat{e}_u = \frac{\frac{\partial \underline{r}}{\partial u}}{\left|\frac{\partial \underline{r}}{\partial u}\right|} = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \qquad \hat{e}_v = \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$$

$$\hat{e}_w = \frac{2w\hat{k}}{2w} = \hat{k}$$

Since $\hat{e}_u \cdot \hat{e}_v = 1 - 1 = 0 = \hat{e}_u \cdot \hat{e}_w = \hat{e}_w \cdot \hat{e}_u$

and $\hat{e}_u \times \hat{e}_v = \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) \times \left(\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right)$

$$= \frac{\hat{k}+\hat{k}}{2} = \hat{k} = \hat{e}_w$$

$$\hat{e}_v \times \hat{e}_w = \frac{1}{\sqrt{2}}(-\hat{i}+\hat{j}) \times \hat{k} = \frac{\hat{i}+\hat{j}}{\sqrt{2}} = \hat{e}_u$$

$$\hat{e}_\omega \times \hat{e}_u = \hat{k} \times \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = \frac{\hat{j}-\hat{i}}{\sqrt{2}} = \hat{e}_v$$

Therefore the given system is orthogonal and right handed. The scale factors are

$$h_u = \left|\frac{\partial \underline{r}}{\partial u}\right| = \sqrt{2} \qquad h_v = \left|\frac{\partial \underline{r}}{\partial v}\right| = \sqrt{2}$$

$$h_\omega = \left|\frac{\partial \underline{r}}{\partial \omega}\right| = 2\omega$$

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(ii) # We know that in orthogonal curvilinear coordinates, the square of the arc length element is given by

$$(ds)^2 = h_u^2 (du)^2 + h_v^2 (dv)^2 + h_\omega^2 (d\omega)^2$$

$$= 2(du)^2 + 2(dv)^2 + 4\omega^2 (d\omega)^2$$

and the volume element is given by

$$dV = h_u h_v h_\omega \, du\, dv\, d\omega$$

$$= (\sqrt{2})(\sqrt{2})(2\omega)\, du\, dv\, d\omega$$

$$= 4\omega \, du\, dv\, d\omega$$

(iii) # Given that $\psi(u,v,\omega) = uv + \omega^2$

$$\Rightarrow \frac{\partial \psi}{\partial u} = v \qquad \frac{\partial \psi}{\partial v} = u \qquad \frac{\partial \psi}{\partial \omega} = 2\omega$$

We know that in an orthogonal curvilinear coordinate system

$$\nabla \psi = \frac{1}{h_u}\frac{\partial \psi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial \psi}{\partial v}\hat{e}_v + \frac{1}{h_\omega}\frac{\partial \psi}{\partial \omega}\hat{e}_\omega$$

$$= \frac{1}{\sqrt{2}} v \hat{e}_u + \frac{1}{\sqrt{2}} u \hat{e}_u + \frac{1}{2\omega}\hat{e}_\omega \cdot 2\omega$$

$$= \frac{1}{\sqrt{2}}(v\hat{e}_u + u\hat{e}_v) + \hat{e}_w$$

(iv) # Given that

$$\underline{A} = wu\hat{e}_u + wv\hat{e}_v + uv\hat{e}_w$$

We know that for orthogonal curvilinear Coordinates

$$\nabla \cdot \underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_u h_w A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right]$$

$$= \frac{1}{(\sqrt{2})(\sqrt{2})(2w)}\left[\frac{\partial}{\partial u}(\sqrt{2}\cdot 2w \cdot wu) + \frac{\partial}{\partial v}(\sqrt{2}\cdot 2w \cdot wv) + \frac{\partial}{\partial w}(\sqrt{2}\cdot\sqrt{2}\cdot uv)\right]$$

$$= \frac{1}{4w}\left[(2\sqrt{2}w^2 + 2\sqrt{2}w^2 + 0)\right]$$

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$$= \frac{1}{4w}(4\sqrt{2}w^2) = \sqrt{2}w$$

and

$$\nabla \times \underline{A} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u A_u & h_v A_v & h_w A_w \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}\cdot\sqrt{2}\cdot 2w}\begin{vmatrix} \sqrt{2}\hat{e}_u & \sqrt{2}\hat{e}_v & 2w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ \sqrt{2}uw & \sqrt{2}wv & 2w\cdot uv \end{vmatrix}$$

$$= \frac{1}{4w}\left[\sqrt{2}\hat{e}_u(2uw - \sqrt{2}v) + \sqrt{2}\hat{e}_v(\sqrt{2}u - 2uw) + 0\right]$$

$$= \frac{1}{2w}\left[(\sqrt{2}uw - u)\hat{e}_u + (u - \sqrt{2}uw)\hat{e}_u\right]$$

(V) # We know that in orthogonal curvilinear Coordinate System

$$\nabla^2\psi = \frac{1}{h_u h_u h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_u h_w}{h_u}\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{h_u h_w}{h_u}\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_u}{h_w}\frac{\partial\psi}{\partial w}\right)\right]$$

$$= \frac{1}{(\sqrt{2})(\sqrt{2})2w}\left[\frac{\partial}{\partial u}\left(\frac{\sqrt{2}\cdot 2w}{\sqrt{2}}\cdot(3u^2)\right) + \frac{\partial}{\partial u}\left(\frac{\sqrt{2}\cdot 2w}{\sqrt{2}}(3u^2)\right) + \frac{\partial}{\partial w}\left(\frac{\sqrt{2}\cdot\sqrt{2}}{2w}(3w^2)\right)\right]$$

$$= \frac{1}{4w}\left(12uw + 12uw + 3\right) \text{ Ans}$$

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Rectangular Coordinates System# .

Problem# Defining rectangular Cartesian Coordinates

(i)# Find Scale factors

(ii) Find Coordinate surfaces (iii) Find Co-ordinate curves (iv) Find expression for arc length, area and Volume elements.

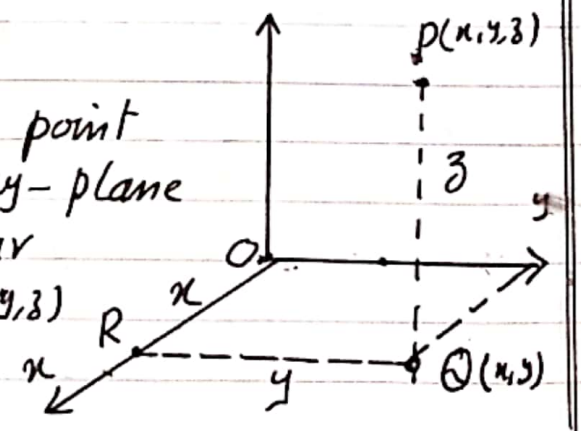
(V) # Find expression for Jacobian

(Vi) # Find expression for gradient, divergence curl and Laplacian

Sol (i) Def.#

Let $P(x,y,z)$ be any point whose projection on the xy-plane is $Q(x,y)$. The rectangular cartesian coordinates (x,y,z) are defined as

$x = OR \quad y = RQ, \quad z = QP$



The unit vectors are $\hat{i}, \hat{j}, \hat{k}$ and any vector $\underline{A}$ in the system can be represented as

$$\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

The position vector $\underline{r}$ in the system is given by

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Scale factors are

$$h_x = h_u = \left|\frac{\partial \underline{r}}{\partial x}\right| = |\hat{i}| = 1$$

$$h_y = h_v = \left|\frac{\partial \underline{r}}{\partial y}\right| = |\hat{j}| = 1$$

$$h_z = h_w = \left|\frac{\partial \underline{r}}{\partial w}\right| = |\hat{k}| = 1$$

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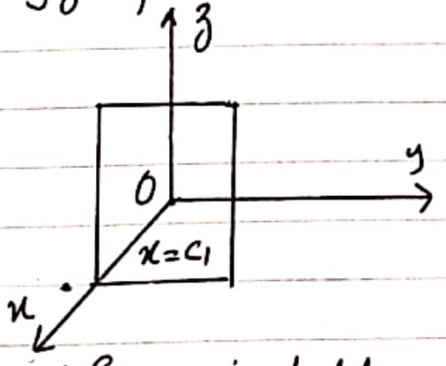
Thus the rectangular Cartesian Coordinate system is a particular case of an orthogonal curvilinear coordinate system where

$u = x \qquad v = y \qquad w = z$

and $h_u = 1 = h_v = h_w$

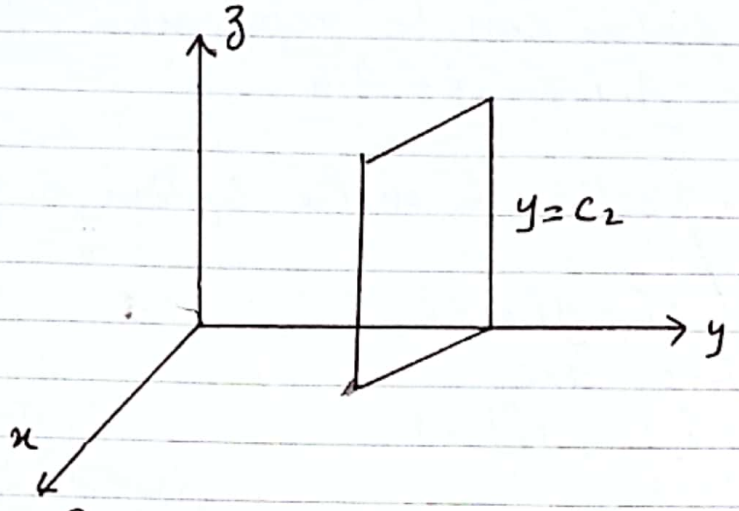
(ii) Coordinate Surfaces

1)# If x is held constant while y & z vary, then the equation $x = c_1$ gives a plane parallel to yz-plane as shown below



2)# If y is held constant while x and z vary, then the equation $y = c_2$ gives a plane parallel to

zx-plane as shown below



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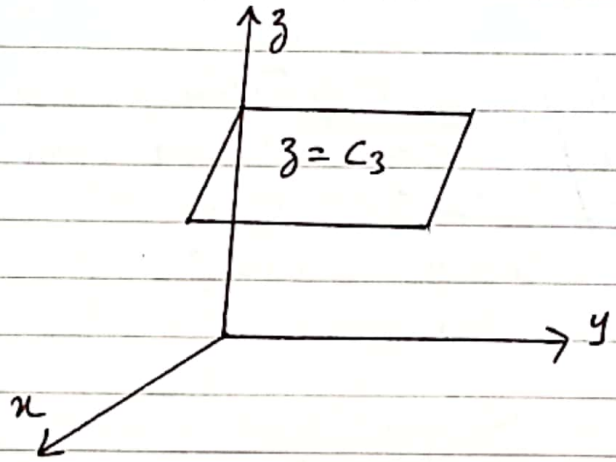
3)# If z is held constant while x & y vary, then the equation $z = c_3$ gives a plane parallel to xy-plane as shown



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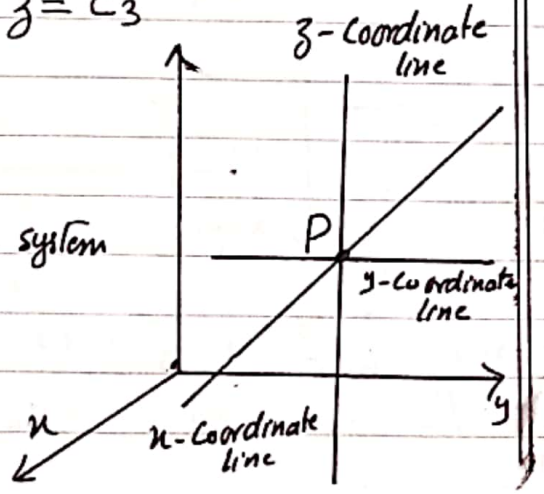
Thus the coordinate surfaces are mutually orthogonal in the sense that any two of them intersect at right angles. Further each point in this system is the intersection of three coordinate surfaces $x = c_1$ $y = c_2$ $z = c_3$

iii)# Coordinate Curves

The coordinate curves for rectangular cartesian coordinate system are

(1) If y and z are fixed and x varies, then $y = c_2$, $z = c_3$ intersect



in a straigh line parallel to x-axis called x-coordinate curve (or line)

(2) # If x and z are fixed while y varies, then the intersection of $x=c_1$ & $z=c_3$ is a straigh line parallel to the y-axis and is called the y-coordinate curve or line

(3) If x and y are fixed while z varies, then the intersection of $x=c_1$ & $y=c_2$ is a straight line. parallel to z-axis and is called the z-coordinate curve (or line)

Thus the coordinate curves of the rectangular Cartesian coordinate system are the straigh lines passing through the point P as shown in fig.

(IV)# Expression For Arc length, Area and Volume Elements#

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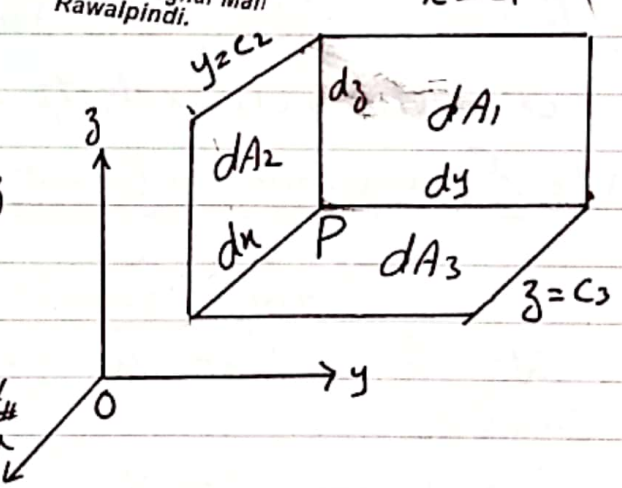

We know that for rectangular Cartesian Coordinates

$u=x \quad v=y \quad w=z$

$h_u = 1 = h_v = h_w$

Arce Length Element#

In orthogonal curvilinear Coordinate system, the element of arc length is determined by

$$(ds)^2 = d\underline{r} \cdot d\underline{r}$$
$$= h_u^2 (du)^2 + h_v^2 (dv)^2 + h_w^2 (dw)^2$$
$$= (1)^2 (dx)^2 + (1)^2 (dy)^2 + (1)^2 (dz)^2$$
$$= (dx)^2 + (dy)^2 + (dz)^2$$

Area Element#

We know that elements of area in an orthogonal curvilinear coordinates

$dA_1 = h_v h_w \, dv\, dw \qquad dA_2 = h_u h_w \, du\, dw$

$dA_3 = h_u h_v \, du\, dv$

In rectangular Cartesian coordinates, these becomes

$dA_1 = (1)(1)\, dy\, dz = dy\, dz$

$dA_2 = (1)(1)\, dx\, dz = dx\, dz$

$dA_3 = (1)(1)\, dx\, dy = dx\, dy$

Volume Element#

We know that the volume element in orthogonal curvilinear coordinates is

$dV = h_u h_v h_w \, du\, dv\, dw$

In rectangular Cartesian coordinates, this becomes

$dV = (1)(1)(1)\, dx\, dy\, dz = dx\, dy\, dz$

(VI)# Expressions for Gradient, Divergence, Curl and Laplacian#

We know that in rectangular Cartesian coordinates

$u = x \qquad v = y \qquad w = z$

$h_u = h_v = h_w = 1$

$\underline{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

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Expression for Gradient#

We know that in Orthogonal Curvilinear Coordinates

$$\nabla \psi = \frac{1}{h_u}\frac{\partial \psi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial \psi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial \psi}{\partial w}\hat{e}_w$$

In rectangular Cartesian Coordinates

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial x}\hat{i} + \frac{1}{1}\frac{\partial\psi}{\partial y}\hat{j} + \frac{1}{1}\frac{\partial\psi}{\partial z}\hat{k}$$

$$= \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$$

Expression for Divergence

In orthogonal curvilinear Coordinates

$$\nabla\cdot A = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial u}(h_u h_w A_u) + \frac{\partial}{\partial w}(h_u h_w A_w)\right]$$

In rectangular Cartesian coordinates.

$$\nabla\cdot A = \frac{1}{(1)(1)(1)}\left[\frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3)\right]$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

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Expression For Curl

In orthogonal curvilinear coordinates, we have

$$\nabla\times\underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(\right.$$

$$= \begin{vmatrix} h_u\hat{e}_u & h_v\hat{e}_v & h_w\hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u A_u & h_v A_v & h_w A_w \end{vmatrix}$$

In rectangular Coordinates

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Expression for Laplacian

In orthogonal curvilinear coordinates, we have

$$\nabla^2\psi = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{h_u h_w}{h_v}\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial \psi}{\partial w}\right)\right]$$

In rectangular Cartesian Coordinates

$$\nabla^2\psi = \frac{1}{(1)(1)(1)}\left[\frac{\partial}{\partial x}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{(1)(1)}{(1)}\frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

(V) Expression for Jacobian

Jacobian in orthogonal curvilinear coordinates is

$$J = h_u h_v h_w$$

In rectangular Cartesian coordinates

$$J = (1)(1)(1) = 1$$

The End of the above Problem

Cylindrical Polar Coordinates

Problem # Defining cylinderical polar coordinates set up a relation between cartesian and cylinderical coordinates and describe coordinate surfaces, co-ordinate curves.

(a) # find unit vectors in cylinderical coordinate system

(b) # find scale factors in this system

(c) # Prove the the cylinderical coordinate system is orthogonal

(d) # find cartesian unit vectors in terms of cylinderical unit vectors

(e) # find a relation between components of vector in

Cartesian and cylinderical polar coordinates

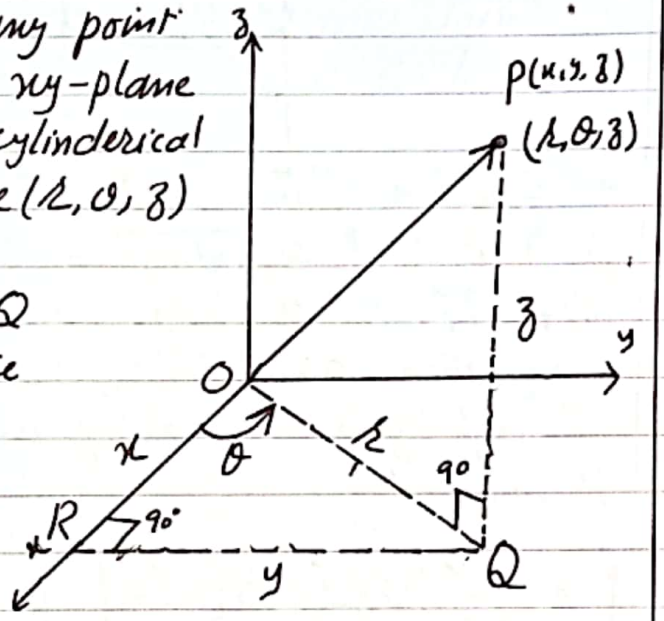
(f) # find expression for arc length, area and volume elements in this system. Also expression for Jacobian

(g) # Derive expression for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in this system

(h) # Find expressions for gradient, divergence, curl and Laplacian

Sol Cylinderical Polar Coordinates (Def)

Let $P(x,y,z)$ be any point whose projection on the xy-plane is $Q(x,y)$. Then the cylinderical coordinates of P are (r,θ,z) in which $r = OQ$ $\theta = \angle XOQ$ and $z = QP$. From the fig the transformation equations expressing the rectangular coordinates in terms of cylinderical coordinates are



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Where $r \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$

Also $x^2+y^2 = r^2$ and $\tan\theta = y/x \Rightarrow \theta = \tan^{-1}(y/x)$

$z = z$

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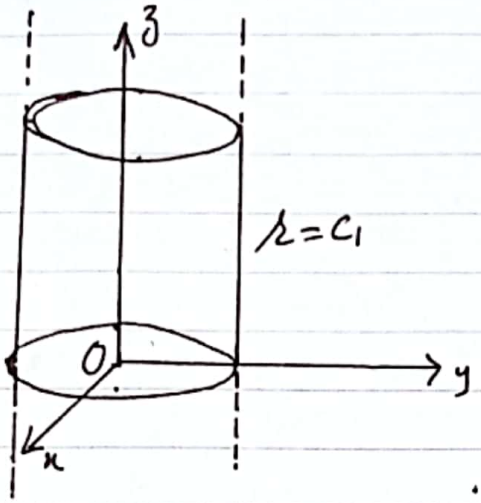
Coordinate Surfaces

In cylinderical coordinate (surfaces) the co-ordinate surfaces are

(i) # If r is held constant while θ and z vary, then the equation $r = c_1$ represents a right circular

of radius c_1 and axis along z-axis (or z-axis if $c_1 = 0$ as shown in fig below



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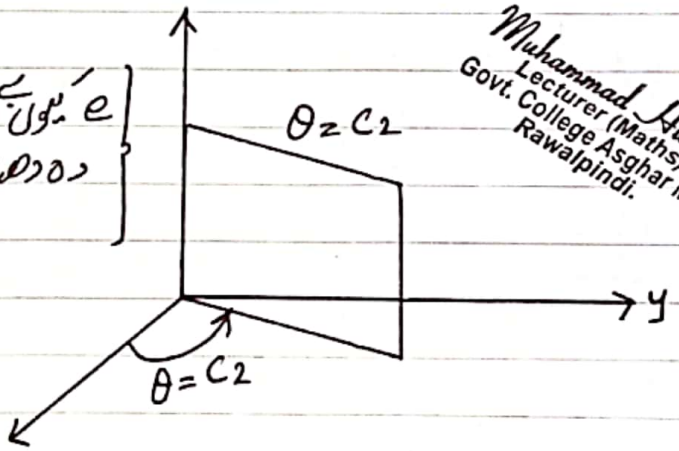
equation of this surface is

$$u = r = \sqrt{x^2+y^2} = c_1$$

(ii) # If θ is held constant and r, z vary, then equation $\theta = \tan^{-1}(y/x) = c_2$ represents half plane through z-axis making angle θ with xz-plane as shown below.

یہ نقطوں پر دخل کرتے ہو مکانوں سے مکینوں کو
وہ دہشت گرد بن جاتے ہیں جن کے گھر نہیں ہوتے
قاسم شاہ



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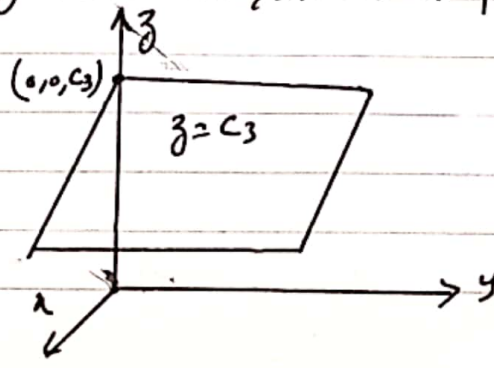
(iii) If z is held constant, while r and θ vary, then equation $z = c_3$ represents a plane perpendicular to z-axis as shown in fig.



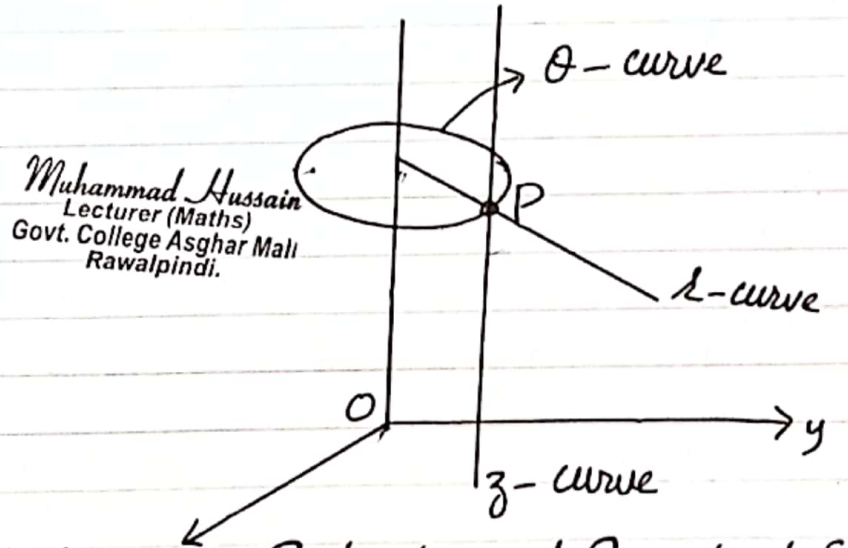
Coordinates Surfaces

The Co-ordinate surfaces for cylinderical polar Coordinates are.

(i) If θ and z are held constant and r varies, then the intersection of $\theta = c_2$ and $z = c_3$ is a straigh line called the r-coordinate curve or r-curve

(ii) # If r and z are fixed while θ varies, then intersection of $r = c_1$ & $z = c_3$ is a circle (or point) called θ-curve

(iii) # If r and θ are fixed and z varies, then intersection of $r = c_1$ & $\theta = c_2$ is a straight line called z-coordinate curve or z-curve.



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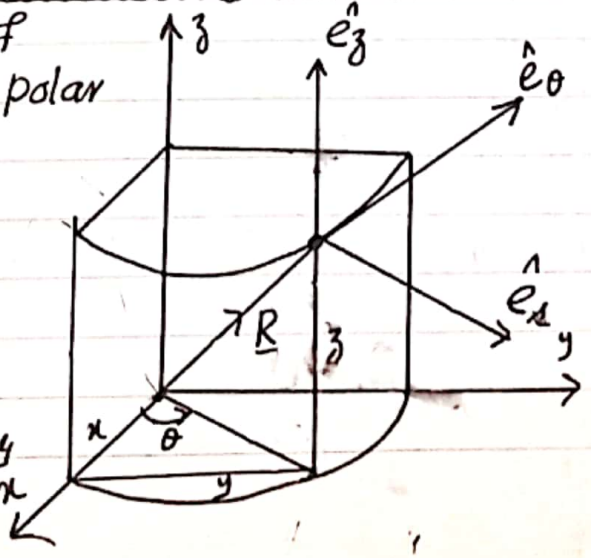
(a) Unit Vectors in Cylinderical Coordinate System

The position vector of any point in cylinderical polar Coordinates is

$$\underline{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}$$

The tangent vectors in the directions of r, θ, z respectively



are

$$\frac{\partial \vec{R}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\frac{\partial \vec{R}}{\partial \theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\frac{\partial \vec{R}}{\partial z} = \hat{k}$$

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The unit vectors in the directions of r, θ, z are given by

$$\hat{e}_r = \frac{\frac{\partial \vec{R}}{\partial r}}{\left|\frac{\partial \vec{R}}{\partial r}\right|} = \frac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \rightarrow (1)$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{R}}{\partial \theta}}{\left|\frac{\partial \vec{R}}{\partial \theta}\right|} = \frac{-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad \rightarrow (2)$$

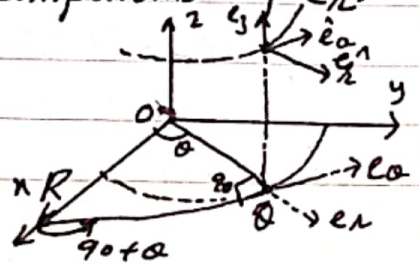
$$\hat{e}_z = \frac{\frac{\partial \vec{R}}{\partial z}}{\left|\frac{\partial \vec{R}}{\partial z}\right|} = \frac{\hat{k}}{|\hat{k}|} = \hat{k} \quad \rightarrow (3)$$

In matrix notation equation (1), (2) & (3) can be written as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ e_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Physically we can write Components of $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_z$ as follows

Angle $\hat{e}_r$ with x-axis θ & with y-axis is $90^\circ - \theta$, with z-axis is 90°



Therefore components of $\hat{e}_r$ are

along x-axis $= \hat{e}_r \cdot \hat{i} = |\hat{e}_r||\hat{i}| \cos\theta = \cos\theta$

along y-axis $= \hat{e}_r \cdot \hat{j} = |\hat{e}_r||\hat{j}| \cos(90-\theta) = \sin\theta$

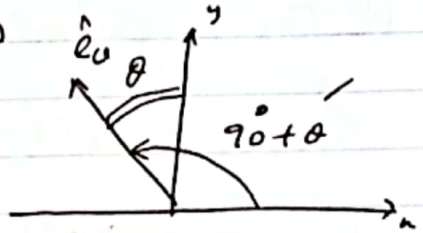
along z-axis $= \hat{e}_r \cdot \hat{k} = |\hat{e}_r||\hat{k}| \cos 90^\circ = 0$

Thus

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\hat{k}$$

From fig $\hat{e}_\theta$ is tar to OQ because tangent and radius at a point of circle are perpendicular

Therefore angle of $\hat{e}_\theta$ with x-axis is $90^\circ + \theta$ and with y-axis is θ, with z-axis is 90°. Hence components of $\hat{e}_\theta$ are



along x-axis $= \hat{e}_\theta \cdot \hat{i}$

$= |\hat{e}_\theta||\hat{i}| \cos(90^\circ + \theta) = -\sin\theta$

along y-axis $= \hat{e}_\theta \cdot \hat{j} = |\hat{e}_\theta||\hat{j}| \cos\theta = \cos\theta$

along z-axis $= \hat{e}_\theta \cdot \hat{k} = |\hat{e}_\theta||\hat{k}| \cos 90^\circ = 0$

$$\Rightarrow \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + 0\hat{k}$$

Angle of $\hat{e}_z$ with x-axis & y-axis is 90° & with z-axis is zero. So Components of $\hat{e}_z$ are

along x-axis $= \hat{e}_z \cdot \hat{i} = |\hat{e}_z||\hat{i}| \cos 90^\circ = 0$

along y-axis $= \hat{e}_z \cdot \hat{j} = |\hat{e}_z||\hat{j}| \cos 90^\circ = 0$

along z-axis $= \hat{e}_z \cdot \hat{k} = |\hat{e}_z||\hat{k}| \cos 0^\circ = \hat{k}$

Hence

$$\hat{e}_z = 0\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k}$$

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Scalar Factors

The scale factors for the cylindrical coordinate system are given by

$$h_u = h_r = \left|\frac{\partial \vec{R}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_r = h_\theta = \left|\frac{\partial \vec{R}}{\partial \theta}\right| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}$$

$$= r\sqrt{\sin^2\theta + \cos^2\theta} = r$$

$$h_\omega = h_z = \left|\frac{\partial \vec{R}}{\partial z}\right| = |\hat{k}| = 1$$

(C)# Orthogonality of Cylinderical Coordinate System#

We know that unit vectors in cylinderical polar coordinates are

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\hat{e}_z = \hat{k}$$

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$$\hat{e}_r \cdot \hat{e}_\theta = (\cos\theta\hat{i} + \sin\theta\hat{j}) \cdot (-\sin\theta\hat{i} + \cos\theta\hat{j})$$

$$= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$\hat{e}_\theta \cdot \hat{e}_z = (-\sin\theta\hat{i} + \cos\theta\hat{j}) \cdot \hat{k} = 0$$

$$\hat{e}_r \cdot \hat{e}_z = (\cos\theta\hat{i} + \sin\theta\hat{j}) \cdot \hat{k} = 0$$

So $\hat{e}_r, \hat{e}_\theta$ and $\hat{e}_z$ are mutually perpendicular and the coordinate system is orthogonal.

(d)# Cartesian unit Vectors in terms of Cylinderical Unit Vectors

We know that

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j} \quad \rightarrow (1)$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j} \quad \rightarrow (2)$$

$$\hat{e}_z = \hat{k} \longrightarrow ③$$

Multiplying ① by $\cos\theta$ and ② by $\sin\theta$ and subtracting, we get

$$\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = (\sin^2\theta + \cos^2\theta)\hat{i}$$

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \longrightarrow ④$$

Multiplying ① by $\sin\theta$ and ② by $\cos\theta$ and then adding

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = (\sin^2\theta + \cos^2\theta)\hat{j}$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta \longrightarrow ⑤$$

Also

$$\hat{k} = \hat{e}_z \longrightarrow ⑥$$

In matrix Equations ④, ⑤ & ⑥ can be written as

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix}$$

OR

Angle between $\hat{i}$ & $\hat{e}_r$ is θ
Angle between $\hat{i}$ & $\hat{e}_\theta$ is $90^\circ + \theta$
Angle between $\hat{j}$ & $\hat{e}_r$ is $90^\circ - \theta$

Component of $\hat{i}$ along $\hat{e}_r$ = $\hat{i}\cdot\hat{e}_r = \cos\theta$
" " of $\hat{i}$ along $\hat{e}_\theta$ = $\hat{i}\cdot\hat{e}_\theta = \cos(90^\circ+\theta)$
Hence $= -\sin\theta$

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

Component of $\hat{j}$ along $\hat{e}_r = \cos(90^\circ - \theta) = \sin\theta$
" " $\hat{j}$ along $\hat{e}_\theta = \cos\theta$
Hence $\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$
$\hat{k} = e_z$

(e) Relation Between Cartesian and Cylinderical Components of a Vector

A vector $\vec{A}$ in rectangular Cartesian coordinates is written as

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} \longrightarrow (1)$$

But $\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$

$\hat{k} = \hat{e}_z$

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using these values in (1)

$$\vec{A} = A_1(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) + A_2(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) + A_3\hat{e}_z$$

$$= (A_1\cos\theta + A_2\sin\theta)\hat{e}_r + (-A_1\sin\theta + A_2\cos\theta)\hat{e}_\theta + A_3\hat{e}_z$$

Thus in Cylinderical system components of $\vec{A}$ are

$$A_r = A_1\cos\theta + A_2\sin\theta$$

$$A_\theta = -A_1\sin\theta + A_2\cos\theta$$

$$A_z = A_3$$

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In matrix form transformation is

$$\begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

A vector $\underline{A}$ in cylindrical polar coordinates can be written as

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_z \hat{e}_z \longrightarrow (2)$$

But

$$\hat{e}_r = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$
$$\hat{e}_\theta = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$$
$$\hat{e}_z = \hat{k}$$

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using these values in (2)

$$\vec{A} = A_r(\cos\theta\, \hat{i} + \sin\theta\, \hat{j}) + A_\theta(-\sin\theta\, \hat{i} + \cos\theta\, \hat{j}) + A_z \hat{k}$$

$$= (A_r \cos\theta - A_\theta \sin\theta)\hat{i} + (A_r \sin\theta + A_\theta \cos\theta)\hat{j} + A_z \hat{k}$$

$$\Rightarrow A_1 = A_r \cos\theta - A_\theta \sin\theta$$

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$$A_2 = A_r \sin\theta + A_\theta \cos\theta$$
$$A_3 = A_z$$

In matrix form transformation is given by

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix}$$

Note # A matrix M is said to be orthogonal if

$$M^t = M^{-1}$$

or if $M^t M = M M^t = I$

Now $\hat{e}_r = \cos\theta\, \hat{i} + \sin\theta\, \hat{j} + 0\hat{k}$

$\hat{e}_\theta = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j} + 0\hat{k}$

$\hat{e}_z = 0\hat{i} + 0\hat{j} + \hat{k}$

In matrix form.

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Here Matrix $M = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

transforms $(\hat{i}, \hat{j}, \hat{k})$ to $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$

Thus Matrix of transformation.

$$(\hat{i}, \hat{j}, \hat{k}) \xrightarrow{T} (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$$

is

$$M = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$M^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$MM^t = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow M$ is orthogonal. This also shows that the transformation is orthogonal. The matrix of inverse transformation

$$(\hat{e}_r, \hat{e}_\theta, \hat{e}_z) \xrightarrow{T^{-1}} (\hat{i}, \hat{j}, \hat{k})$$

is $$M^t = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This formation help us to memorize the Matrices these transformation because these matrics are transpose of each other.

(f) # Expression for arc Length, area & Volume Elements in Cylindical Polar Coordinates

In cylinderical polar coordinates we have

$u = r \quad v = \theta \quad w = z$

$h_u = 1 \quad h_v = r \quad h_w = 1$

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Arc length Element

In orthogonal curvilinear Coordinates the arc length element is determined from.

$$(ds)^2 = h_u^2(du)^2 + h_v^2(dv)^2 + h_w^2(dw)^2$$

In cylinderical polar coordinates, this is

$$(ds)^2 = (1)^2(dr)^2 + r^2(d\theta)^2 + (1)^2(dz)^2$$

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

Alternate Method (Direct Method)

In cylinderical polar coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$x = x(r, \theta)$ $\quad y = y(r, \theta)$ $\quad z = z(z)$

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta dr + r\cos\theta d\theta$$

$$dz = dz$$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 + (dz)^2$$

$$= (\cos^2\theta + \sin^2\theta)(dr)^2 + r^2(\cos^2\theta + \sin^2\theta)(d\theta)^2 + (dz)^2$$

$$= (dr)^2 + (r d\theta)^2 + (dz)^2$$

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Area Elements

We know that the elements of area in orthogonal curvilinear coordinates are given by

$dA_1 = h_u h_w \, du\, dw$ area bounded by $v-$, $w-$ curves

$dA_2 = h_u h_w \, du\, dw$ area bounded by $u-$, $w-$ curves

$dA_3 = h_u h_v \, du\, dv$ area bounded by $u-$ $v-$ curves

In cylindrical polar coordinates, these are

$dA_1 = (r)(1)\, d\theta\, dz = r\, d\theta\, dz$ area bounded by θ & z-curves

$dA_2 = (1)(1)\, dr\, dz = dr\, dz$

$dA_3 = (1)(r)\, dr\, d\theta = r\, dr\, dz$

Volume Element

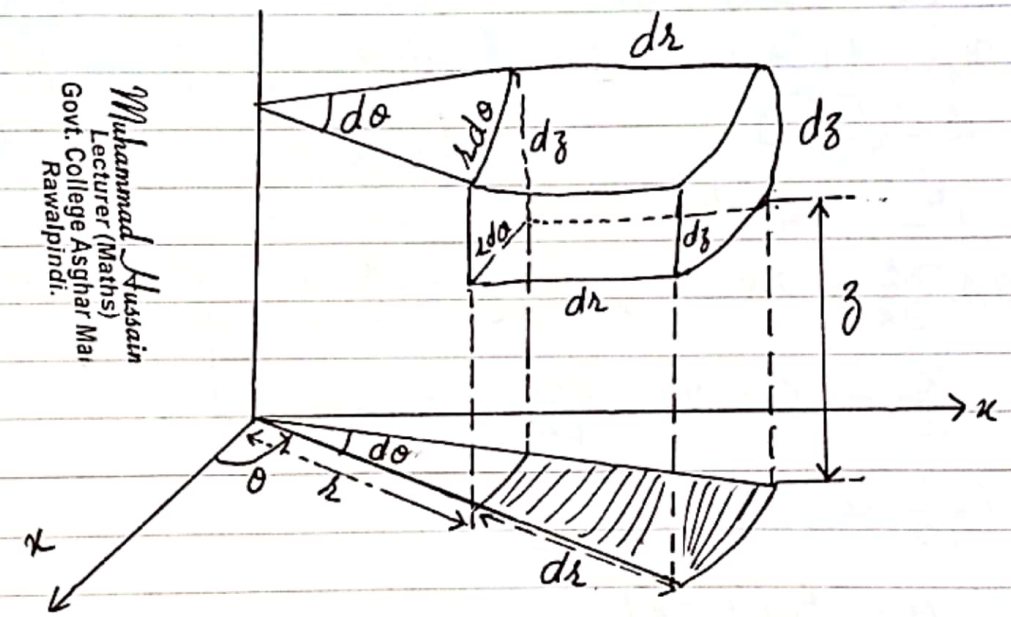
The volume element in orthogonal curvilinear coordinates is

$$dV = h_u h_v h_w \, du\, dv\, dw$$

In cylinderical polar coordinates

$$dV = (1)(r)(1)\, dr\, d\theta\, dz$$
$$= r\, dr\, d\theta\, dz$$

Sketch of Volume element is



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Edges of volume elements shown above are.

dr, $r\,d\theta$, dz

so Volume $= dV = r\, dr\, d\theta\, dz$

Expression for Jacobian in Cylinderical Polar Coordinates

We know that the Jacobian in orthogonal curvilinear coordinates u, v, w is given by

$$J = J\left(\frac{x,y,z}{u,v,w}\right) = h_u h_v h_w$$

In cylindrical coordinates $h_u = 1$ $h_v = r$
$h_w = 1$ $u = r$ $v = \theta$ $z = z = w$

Therefore $J\left(\frac{x,y,z}{r,\theta,z}\right) = (1)(r)(1) = r$

(9) Expression for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in cylindrical Polar Coordinates

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We know that

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$

$r^2 = x^2 + y^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$

$\frac{x}{r} = \cos\theta = \frac{\partial r}{\partial x}$ $\longrightarrow$ (1)

$\frac{\partial r}{\partial y} = y/r = \frac{r\sin\theta}{r} = \sin\theta$ $\longrightarrow$ (2)

$\theta = \tan^{-1}(y/x)$

$\Rightarrow \frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)$

$= -\frac{y}{r^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$

$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$ $\longrightarrow$ (3)

$\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$ $\longrightarrow$ (4)

$\frac{\partial z}{\partial x} = 0$ $\frac{\partial z}{\partial y} = 0$ $\frac{\partial z}{\partial z} = 1$ $\longrightarrow$ (5)

Now by chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x}$$

$$= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \longrightarrow (6)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y}$$

$$= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \longrightarrow (7)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial z}$$

$$= (0)\frac{\partial}{\partial r} + (0)\frac{\partial}{\partial \theta} + (1)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \longrightarrow (8)$$

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(6), (7) & (8) are required expressions in terms of cylinderical polar coordinates.

(h) # Expression for Gradient, Divergence, Curl and Laplacian

We know that for cylinderical polar coordinates (r, θ, z)

$u = r \quad v = \theta \quad w = z \quad \hat{e}_u = \hat{e}_r \quad \hat{e}_v = \hat{e}_\theta \quad \hat{e}_w = \hat{e}_z$

$h_u = 1 \quad h_v = r \quad h_w = 1$

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Expression for Gradient

In orthogonal curvilinear coordinates

$$\nabla\psi = \frac{1}{h_u}\frac{\partial\psi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial\psi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial\psi}{\partial w}$$

In cylinderical polar coordinates.

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{1}\frac{\partial\psi}{\partial z}\hat{e}_z$$

Alternate Method

We know that in cylindrical coordinates

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

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$$\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \qquad \hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$$

$$\hat{k} = \hat{e}_z$$

we have $\nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$ $\longrightarrow$ (1)

Now

$$\frac{\partial\psi}{\partial x}\hat{i} = \left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)\left(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta\right)$$

$$= \cos^2\theta\frac{\partial\psi}{\partial r}\hat{e}_r - \frac{\sin\theta\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_r - \sin\theta\cos\theta\frac{\partial\psi}{\partial r}\hat{e}_\theta + \frac{\sin^2\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta \qquad \longrightarrow (2)$$

$$\frac{\partial\psi}{\partial y}\hat{j} = \left(\sin\theta\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\right)\left(\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta\right)$$

$$= \sin^2\theta\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{\sin\theta\cos\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_r + \frac{\cos^2\theta}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \sin\theta\cos\theta\frac{\partial\psi}{\partial r}\hat{e}_\theta \qquad \longrightarrow (3)$$

$$\frac{\partial\psi}{\partial z}\hat{k} = \frac{\partial\psi}{\partial z}\hat{e}_z \qquad \longrightarrow (4)$$

251+2 = 253

Adding ②, ③ and ④

$$\frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j} + \frac{\partial \psi}{\partial z}\hat{k}$$

$$= (\cos^2\theta + \sin^2\theta)\frac{\partial \psi}{\partial r}\hat{e}_r + \left(\frac{\sin^2\theta + \cos^2\theta}{r}\right)\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z$$

$$= \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z$$

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Expression For Divergence

In orthogonal curvilinear coordinates, we have

$$\nabla \cdot \underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_u h_w A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right]$$

In cylinderical polar Coordinates, this is

$$\nabla \cdot \underline{A} = \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}(r A_r) + \frac{\partial}{\partial \theta}(1)(1) A_\theta) + \frac{\partial}{\partial z}(1)(r) A_z)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_r) + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial}{\partial z}(r A_z)\right]$$

Alternate Method

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$$A_1 = A_r \cos\theta - A_\theta \sin\theta$$

$$A_2 = A_r \sin\theta + A_\theta \cos\theta$$

$$A_3 = A_3$$

We know that

$$\nabla \cdot \underline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\frac{\partial A_1}{\partial x} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)(A_r \cos\theta - A_\theta \sin\theta)$$

$$= \cos^2\theta \frac{\partial A_r}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial A_r}{\partial \theta} + \frac{\sin^2\theta}{r} A_r$$

$$- \sin\theta\cos\theta \frac{\partial A_\theta}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\sin\theta\cos\theta}{r} A_\theta \longrightarrow (1)$$

$$\frac{\partial A_r}{\partial y} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)(A_r \sin\theta + A_\theta \cos\theta)$$

$$= \sin^2\theta \frac{\partial A_r}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial A_r}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial A_\theta}{\partial \theta}$$

$$+ \frac{\cos^2\theta}{r} A_r + \left(\frac{\cos^2\theta}{r}\frac{\partial A_\theta}{\partial \theta}\right) + \sin\theta\cos\theta \frac{\partial A_\theta}{\partial r} \longrightarrow (2)$$

$$\frac{\partial A_3}{\partial z} = \frac{\partial A_3}{\partial z} \longrightarrow (3)$$

Adding (1) & (2) & (3)

$$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = (\cos^2\theta + \sin^2\theta)\frac{\partial A_r}{\partial r} + \frac{1}{r}(\sin^2\theta + \cos^2\theta) A_r + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

$$\nabla \cdot \underline{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_3}{\partial z}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_r) + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial}{\partial z}(r A_z)\right]$$

Expression For Curl

In orthogonal curvilinear coordinates, we have

$$\nabla \times \underline{A} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u A_u & h_v A_v & h_w A_w \end{vmatrix}$$

$\underline{\underline{25}}3+2 = \underline{255}$

In cylinderical polar coordinates.

$$\nabla \times \underline{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z}(rA_\theta)\right)\hat{e}_r + \left(r\frac{\partial A_r}{\partial z} - r\frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{e}_z\right]$$

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Expression For Laplacian

In orthogonal curvilinear coordinates

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$$\nabla^2\psi = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial \psi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial \psi}{\partial w}\right)\right]$$

In (r, θ, z)

$$\nabla^2\psi = \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2} + r\frac{\partial^2 \psi}{\partial z}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z}$$

$$= \frac{1}{r}\left[\frac{\partial \psi}{\partial r} + r\frac{\partial^2 \psi}{\partial r^2}\right] + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z}$$

$$= \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z}$$

$$= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z}$$

★ Bertrand Russel said. "work in every case is a great blessing of God & it should be applied to acquisition of pleas

Alternate Method

We know that

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z \longrightarrow (1)$$

$$\nabla\cdot\underline{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \longrightarrow (2)$$

From (1) & (2)

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$

is the Laplacian operator.

Problem# Transform $\underline{A} = x\hat{i} + y\hat{j} + z\hat{k}$ in Circular cylinerical coordinates

Sol # Here $A_1 = x \quad A_2 = y \quad A_3 = z$

$$A_r = A_1\cos\theta + A_2\sin\theta = x\cos\theta + y\sin\theta$$

$$A_\theta = -A_1\sin\theta + A_2\cos\theta = -x\sin\theta + y\cos\theta$$

$$A_z = A_3 = z$$

putting $x = r\cos\theta \quad y = r\sin\theta$

$$A_r = r\cos^2\theta + r\sin^2\theta = r$$

$$A_\theta = -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0$$

$$A_z = z$$

$\underline{A}$ in cylinderical polar coordinates is

$$\underline{A} = r\hat{e}_r + z\hat{k}$$

Problem# Transform $\underline{A} = \frac{1}{r}\hat{e}_r$ into rectangular cartesian coordinates.

Sol # $A_\theta = 0 \quad A_r = \frac{1}{r} \quad A_z = 0$

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$A_1 = A_r \cos\theta - A_\theta \sin\theta = \frac{1}{r}\cos\theta$

$A_2 = A_r \sin\theta + A_\theta \cos\theta = \frac{1}{r}\sin\theta$

$A_3 = A_z = 0$

putting $r = \sqrt{x^2+y^2}$ $\cos\theta = x/r$

$\sin\theta = y/r$

$A_1 = \frac{x}{x^2+y^2}$ $A_2 = \frac{y}{x^2+y^2}$ $A_3 = 0$

Hence $\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$= \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j} + 0\hat{k}$

Problem Transform the relation $\underline{A} = (x/y)\hat{i}$ into cylinderical coordinates.

Sol # $A_1 = x/y$ $A_2 = 0$ $A_3 = 0$

$A_r = A_1\cos\theta + A_2\sin\theta$

$= x/y \cos\theta$

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$A_\theta = -A_1\sin\theta + A_2\cos\theta = -x/y \sin\theta$

$A_z = A_3 = 0$

putting $x = r\cos\theta$ $y = r\sin\theta$

$A_r = \left(\frac{r\cos\theta}{r\sin\theta}\right)\cos\theta = = \cot\theta\cos\theta$

$A_\theta = -\frac{r\cos\theta}{r\sin\theta}\cdot\sin\theta = -\cos\theta$

Therefore

$\underline{A} = A_r\hat{e}_\theta + A_\theta\hat{e}_\theta + A_z\hat{e}_3$

$= \cot\theta\cos\theta\,\hat{e}_r - \cos\theta\,\hat{e}_\theta$

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Spherical Polar Coordinates

Problem (a) Defining spherical polar coordinates set up a relation between cartesian and spherical coordinates. Describe coordinate surfaces and curves

(b) # Find unit vectors in spherical coordinates

(c) # Find scale factors in the system

(d) # Prove that the spherical coordinate system is orthogonal

(e) # Find relation between cartesian unit vectors and spherical unit vectors (cartesian interms of spherical)

(f) # Find a relation between cartesian and spherical components of a Vector

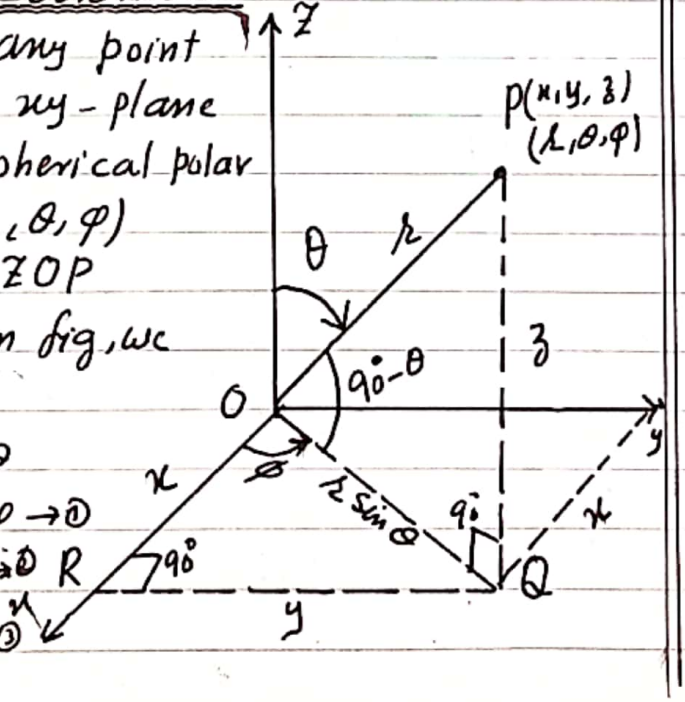
(g) # Find expressions for arc length, area, Volume and Jacobian

(h) # Derive expressions for $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ in spherical polar coordinates

(i) # Find expressions for gradient, divergence, curl and Laplacian in spherical coordinates.

Sol #(a) Spherical Polar Coordinates

Let $P(x,y,z)$ be any point whose projection on the xy-plane is $Q(x,y)$. Then the spherical polar coordinates of P are (r,θ,φ) in which $r = OP$ $\theta = \angle ZOP$ and $\varphi = \angle XOQ$. From fig, we have



$OQ = r\cos(90°-\theta) = r\sin\theta$

$x = OQ\cos\varphi = r\sin\theta\cos\varphi \rightarrow (1)$

$y = OQ\sin\varphi = r\sin\theta\sin\varphi \rightarrow (2)$

$z = OP\sin(90°-\theta) = r\cos\theta \rightarrow (3)$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$

squaring and adding ①, ② & ③

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta$$
$$= r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi) + r^2\cos^2\theta$$
$$= r^2\sin^2\theta + r^2\cos^2\theta$$
$$= r^2$$
$$r = \sqrt{x^2+y^2+z^2}$$

squaring and adding ① & ②

$$x^2+y^2 = r^2\sin^2\theta(\cos^2\varphi+\sin^2\varphi)$$
$$= r^2\sin^2\theta$$
$$r\sin\theta = \sqrt{x^2+y^2} \longrightarrow ④$$

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Dividing ④ by ③

$$\frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{x^2+y^2}}{z} \quad \text{or} \quad \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

Dividing ② by ①

$$\frac{r\sin\theta\sin\varphi}{r\sin\theta\cos\varphi} = y/x \quad \text{or} \quad \tan\varphi = y/x$$
$$\Rightarrow \varphi = \tan^{-1}(y/x)$$

Hence the equations giving spherical polar co-ordinates in terms of cartesian coordinates

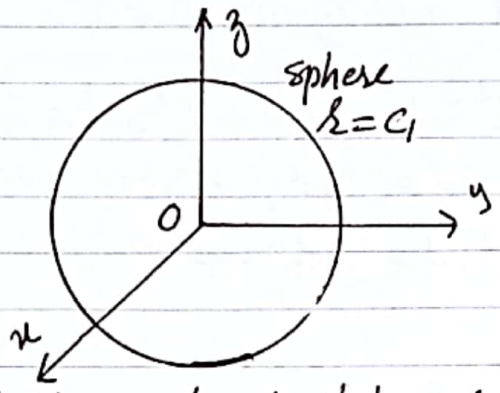
$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad \varphi = \tan^{-1}(y/x)$$

Coordinate Surfaces#

In spherical coordinate system, the coordinate surfaces are

(i) If r is held constant while θ and φ vary, then equation $r = c_1$ gives a sphere with centre at

at the origin (or origin if $c_1 = 0$) as shown



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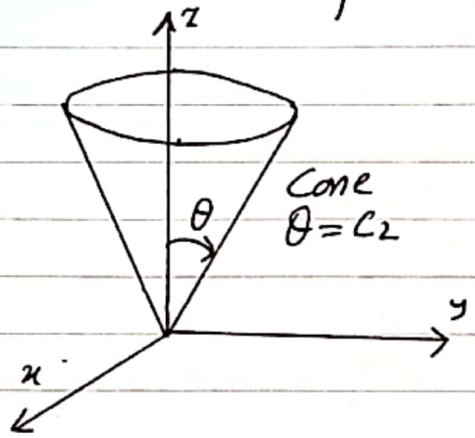
(ii) # If θ is held constand while ρ and φ vary, then the equation $\theta = c_2$ represents cone as shown



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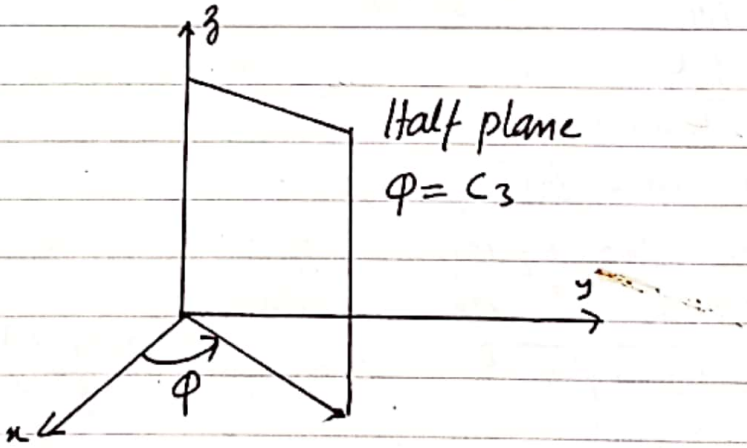
(iii) If φ is held constant while ρ and θ vary, then equation $\varphi = c_3$ represent a half-plane through z-axis as shown



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Coordinate Curves

The coordinate curves for spherical polar coordinates are.

(i) # If θ and φ are fixed and ρ varies then the

the intersection of $\theta = c_2$ & $\varphi = c_3$ is a straight line called the r- coordinate curve or r-curve

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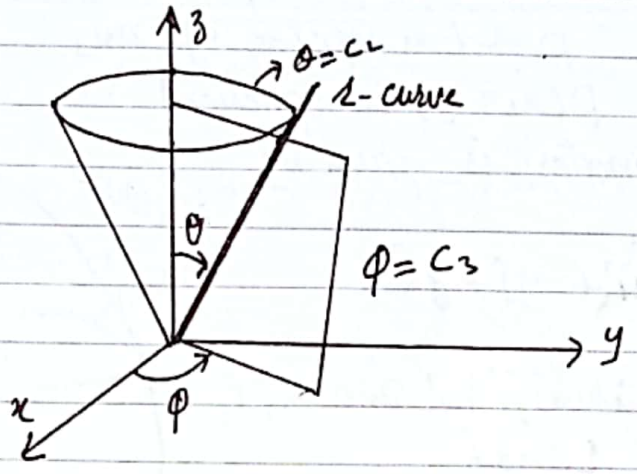

(ii) If r and φ are fixed while θ varies, then intersection of $r = c_1$ and $\varphi = c_3$ is a semi-circle ($c_1 \neq 0$) called θ- coordinate curve or θ curve

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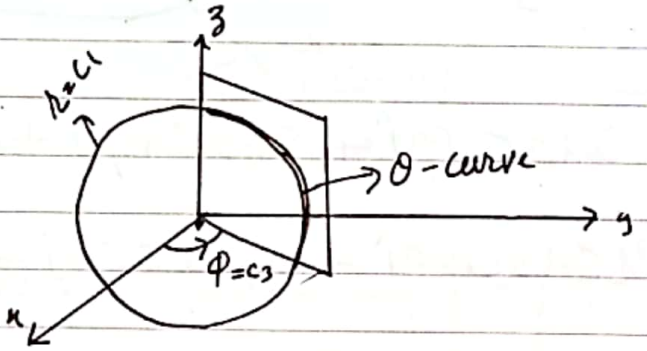

(iii) If r and θ are fixed while φ varies, then intersection of $r = c_1$ and $\theta = c_2$ is a circle (or a point) called φ- coordinate curve or φ curve

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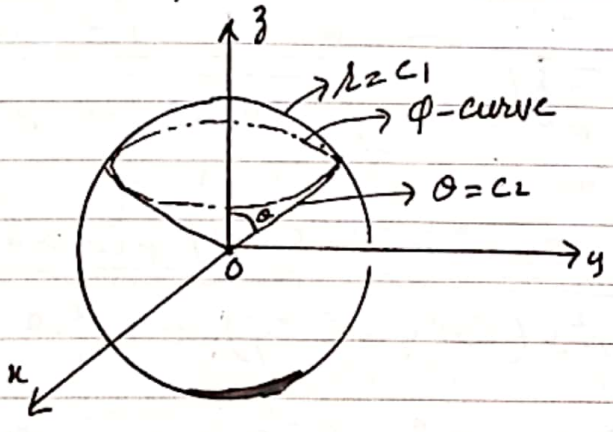
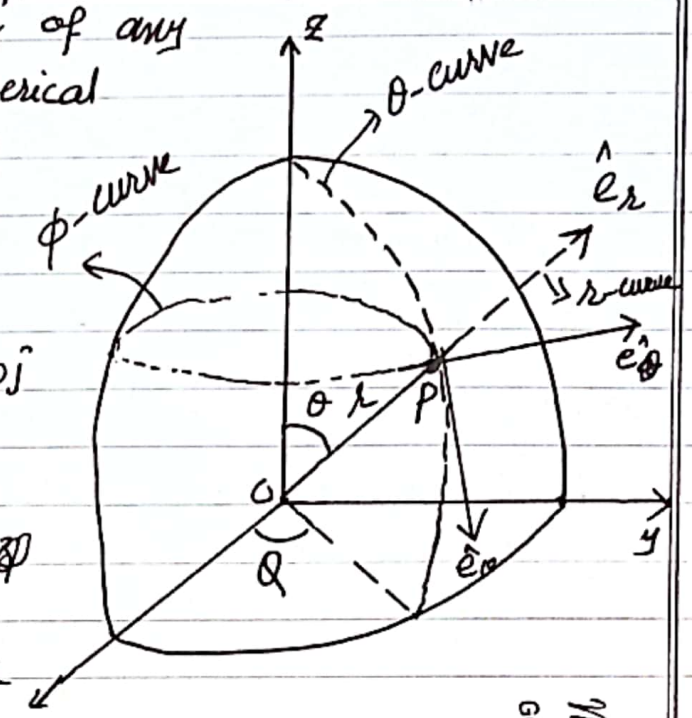


Unit Vectors in Spherical Coordinate System#

The position vector of any point $P(x,y,z)$ in spherical coordinates is given by



$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\sin\theta\cos\varphi\hat{i} + r\sin\theta\sin\varphi\hat{j} + r\cos\theta\hat{k}$$

The tangent vectors in the directions of r, θ, φ are given by

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}$$

$$\frac{\partial \underline{r}}{\partial \theta} = +r\cos\theta\cos\varphi\hat{i} + r\cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}$$

$$\frac{\partial \underline{r}}{\partial \varphi} = -r\sin\theta\sin\varphi\hat{i} + r\sin\theta\cos\varphi\hat{j} + 0\hat{k}$$

The unit vectors in the directions of r, θ, φ are

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{\sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}}{\sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}}$$

$$= \frac{\sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}}{\sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta}}$$

$$\hat{e}_r = \sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k} \longrightarrow (1)$$

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$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\varphi\,\hat{i} + r\cos\theta\sin\varphi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}}$$

$$= \frac{r\cos\theta\cos\varphi\,\hat{i} + r\cos\theta\sin\varphi\,\hat{j} - r\sin\theta\,\hat{k}}{r}$$

$$= \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k} \longrightarrow (2)$$

$$\hat{e}_\varphi = \frac{\frac{\partial \vec{r}}{\partial \varphi}}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} = \frac{-r\sin\theta\sin\varphi\,\hat{i} + r\sin\theta\cos\varphi\,\hat{j}}{\sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}}$$

$$= -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \longrightarrow (3)$$

In matrix notation equation (1), (2) & (3) can be written as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

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(C) **Scale factors #**

The scale factors for spherical coordinate system is

$$h_u = h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$h_v = h_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta}$$

$$= r$$

$$h_w = h_\varphi = \frac{\partial r}{\partial \varphi} \Big/ \left|\frac{\partial r}{\partial \varphi}\right| = \sqrt{r^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi)}$$

$h_\varphi = r\sin\theta$

Hence the scale factors are

$h_u = h_r = 1 \quad h_v = h_\theta = r \quad h_w = h_\varphi = r\sin\theta$

(d) # Orthogonality of Spherical Coordinates

We know that unit vectors in spherical polar coordinates are

$\hat{e}_r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

$\hat{e}_r \cdot \hat{e}_\theta = (\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}) \cdot (\cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k})$

$= \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta$

$= \sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi) - \sin\theta\cos\theta$

$= \sin\theta\cos\theta - \sin\theta\cos\theta = 0$

$\hat{e}_r \cdot \hat{e}_\varphi = (\sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}) \cdot (-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})$

$= -\sin\theta\cos\varphi\sin\varphi + \sin\theta\sin\varphi\cos\varphi = 0$

$\hat{e}_\theta \cdot \hat{e}_\varphi = (\cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}) \cdot (-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})$

$= -\cos\theta\sin\varphi\cos\varphi + \cos\theta\sin\varphi\cos\varphi + 0 = 0$

So $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ are mutually perpendicular and the coordinate system is orthogonal

(d) Cartesian Unit Vectors in terms of Spherical Unit Vectors

We know that the unit vectors in spherical polar co-ordinates are

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k} \longrightarrow (1)$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k} \longrightarrow (2)$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} + 0\,\hat{k} \longrightarrow (3)$$

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and then adding

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = \cos\varphi(\sin^2\theta+\cos^2\theta)\hat{i} + \sin\varphi(\sin^2\theta+\cos^2\theta)\hat{j}$$

$$\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \longrightarrow (4)$$

Multiplying (4) by $\cos\varphi$ and (3) by $\sin\varphi$ and subtracting

$$\sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi = (\cos^2\varphi+\sin^2\varphi)\hat{i}$$

$$\Rightarrow \hat{i} = \sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi \longrightarrow (5)$$

Similarly multiply (4) by $\sin\varphi$ and (3) by $\cos\varphi$ and adding we get

$$\sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi = (\sin^2\varphi+\cos^2\varphi)\hat{j}$$

$$\hat{j} = \sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi \longrightarrow (6)$$

Multiplying (1) by $\cos\theta$ and (2) $\sin\theta$ and subtracting

$$\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = (\cos^2\theta+\sin^2\theta)\hat{k}$$

$$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta \longrightarrow (7)$$

In matrix notation (5), (6) & (7) can be written as

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix}$$

OR

Components of $\hat{i}$ along $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are

Compt. along $\hat{e}_r = \hat{i}\cdot\hat{e}_r = \sin\theta\cos\phi$

Compnt along $\hat{e}_\theta = \hat{i}\cdot\hat{e}_\theta = \cos\theta\cos\phi$

Compnt along $\hat{e}_\phi = i\cdot\hat{e}_\phi = -\sin\phi$

Hence

$$\hat{i} = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$$

Compnt of $\hat{j}$ along $e_r = \hat{j}\cdot\hat{e}_r = \sin\theta\sin\phi$

" " $\hat{j}$ " $e_\theta = \hat{j}\cdot\hat{e}_\theta = \cos\theta\sin\phi$

" " $\hat{j}$ " $e_\phi = \hat{j}\cdot\hat{e}_\phi = \cos\phi$

Hence

$$\hat{j} = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$$

Compnt of $\hat{k}$ along $\hat{e}_r = \hat{k}\cdot\hat{e}_r = \cos\theta$

" " $\hat{k}$ " $\hat{e}_\theta = \hat{k}\cdot\hat{e}_\theta = -\sin\theta$

" " $\hat{k}$ " $\hat{e}_\phi = \hat{k}\cdot\hat{e}_\phi = 0$

Hence

$$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

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Note# note that the unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ vary in direction from point remaining mutually perpendicular just like a solid frame while the unit vectors $\hat{i}, \hat{j}, \hat{k}$ maintain their fixed directions.

(e) Relation Between Components of a Vector in Cartesian and Spherical Coordinates:

A vector $\underline{A}$ in rectangular cartesian coordinates can be written as

$$\underline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

putting values of $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$

$$\underline{A} = A_1\left(\sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi\right)$$
$$+ A_2\left(\sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi\right)$$
$$+ A_3\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)$$
$$= \left[A_1\left(\sin\theta\cos\varphi\right) + A_2\sin\theta\sin\varphi + A_3\cos\theta\right]\hat{e}_r$$
$$+ \left(A_1\cos\theta\cos\varphi + A_2\cos\theta\sin\varphi - A_3\sin\theta\right)\hat{e}_\theta$$
$$+ \left(-A_1\sin\varphi + A_2\cos\varphi\right)\hat{e}_\varphi$$

$$\Rightarrow A_r = A_1\sin\theta\cos\varphi + A_2\sin\theta\sin\varphi + A_3\cos\theta \quad \rightarrow (1)$$

$$A_\theta = A_1\cos\theta\cos\varphi + A_2\cos\theta\sin\varphi - A_3\sin\theta \quad \rightarrow (2)$$

$$A_\varphi = -A_1\sin\varphi + A_2\cos\varphi \quad \rightarrow (3)$$

In matrix notation we have from (1), (2) & (3)

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Similarly A vector $\underline{A}$ in spherical polar co-ordinates can be expressed as

$$\underline{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\varphi\hat{e}_\varphi$$

putting values of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ in terms of $\hat{i}, \hat{j}, \hat{k}$

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$$\underline{A} = A_r(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}) + A_\theta(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}) + A_\phi(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$= (A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi)\hat{i} + (A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi)\hat{j} + (A_r\cos\theta - A_\theta\sin\theta)\hat{k}$$

$$\Rightarrow A_1 = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_2 = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_3 = A_r\cos\theta - A_\theta\sin\theta$$

In matrix form

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

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(f) Expressions for Arc length, area, Volume Elements and Laplacian in Spherical System#

In spherical polar coordinates, we have

$u = r \quad v = \theta \quad w = \phi$

$\hat{e}_u = \hat{e}_r \quad \hat{e}_v = \hat{e}_\theta \quad \hat{e}_w = \hat{e}_\phi$

$h_u = h_r = 1 \quad h_v = h_\theta = r \quad h_w = h_\phi = r\sin\theta$

Arc length Element#

In spherical polar co-ordinates, we have

For orthogonal curvilinear coordinates, we have

$$(ds)^2 = h_u^2(du)^2 + h_v^2(dv)^2 + h_w^2(dw)^2$$

In spherical polar coordinates

$$(ds)^2 = (1)^2(dr)^2 + (r)^2(d\theta)^2 + (r\sin\theta)^2(d\varphi)^2$$

$$= (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2$$

Alternate Method

In spherical polar Co-ordinates

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

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$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi$$

$$dx = \sin\theta\cos\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + (-r\sin\theta\sin\varphi)\,d\varphi \rightarrow (1)$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \varphi}d\varphi$$

$$= \sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + r\sin\theta\cos\varphi\,d\varphi \rightarrow (2)$$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \varphi}d\varphi$$

$$= \cos\theta\,dr - r\sin\theta\,d\theta + 0 \rightarrow (3)$$

squaring and adding (1), (2) & (3)

$$(dx)^2 + (dy)^2 + (dz)^2 = \sin^2\theta\cos^2\varphi(dr)^2 + \sin^2\theta\sin^2\varphi(dr)^2 + \cos^2\theta(dr)^2$$
$$+ r^2\cos^2\theta\sin^2\varphi(d\theta)^2 + r^2\cos^2\theta\sin^2\varphi(d\theta)^2 + r^2\sin^2\theta(d\theta)^2$$
$$+ 2r\sin\theta\cos\theta\sin\varphi\cos\varphi\,dr\,d\theta - 2r^2\sin\theta\cos\theta\sin\varphi\cos\varphi(d\theta)\,d\varphi$$
$$- 2r\sin^2\theta\sin\varphi\cos\varphi\,dr\,d\varphi + 2r\sin\theta\cos\theta\sin^2\varphi\,dr\,d\theta$$

$$+2r\sin^2\theta\sin\phi\cos\phi\,dr\,d\phi + 2r^2\sin\theta\cos\theta\sin\phi\cos\phi\,d\theta\,d\phi - 2r\sin\theta\cos\theta\,dr\,d\theta$$

$$= [\sin^2\theta(\cos^2\phi+\sin^2\phi)+\cos^2\theta](dr)^2 + [r^2\sin^2\theta(\sin^2\phi+\cos^2\phi)](d\phi)^2 + [r^2\cos^2\theta(\cos^2\phi+\sin^2\phi)+r^2\sin^2\theta](d\theta)^2 + [2r\sin\theta\cos\theta(\cos^2\phi+\sin^2\phi)-2r\sin\theta\cos\theta]\,dr\,d\theta$$

$$= (\sin^2\theta+\cos^2\theta)(dr)^2 + r^2\sin^2\theta(d\phi)^2 + r^2(d\theta)^2$$

$$= (dr)^2 + (r\,d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

Area Element

We know that elements of area in orthogonal curvilinear coordinates are

$dA_1 = h_u h_v\,du\,dv$ =

$dA_2 = h_u h_w\,du\,dw$ $\quad dA_3 = h_u h_v\,du\,dv$

In spherical polar co-ordinates, these becomes

$$dA_1 = (r)(r\sin\theta)\,d\theta\,d\phi = r^2\sin\theta\,d\theta\,d\phi$$

$$dA_2 = (1)(r\sin\theta)\,dr\,d\phi = r\sin\theta\,dr\,d\phi$$

$$dA_3 = (1)(r)\,dr\,d\theta = r\,dr\,d\theta$$

Volume Element

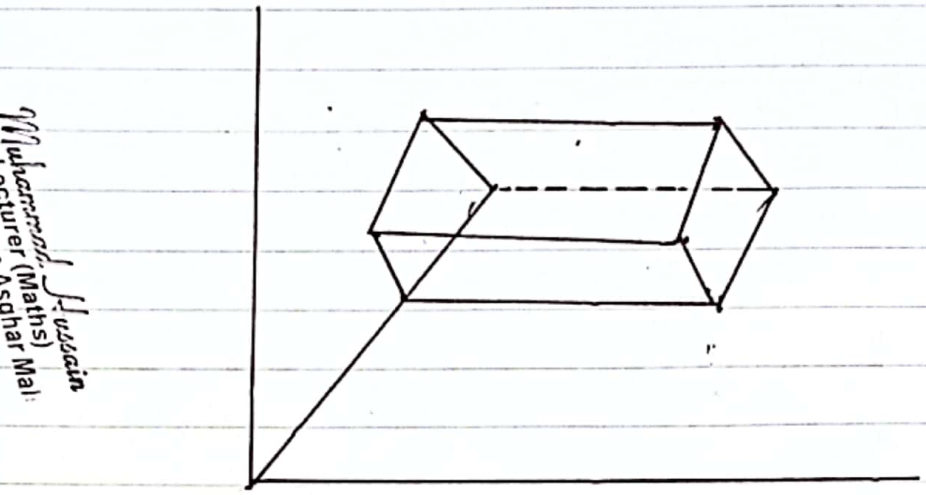
The volume element in orthogonal curvilinear co-ordinates is

$dV = h_u h_v h_w\,du\,dv\,dw$

In spherical polar co-ordinates, this is

$$dV = (1)(r)(r\sin\theta)\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

The sketch of volume element is given below



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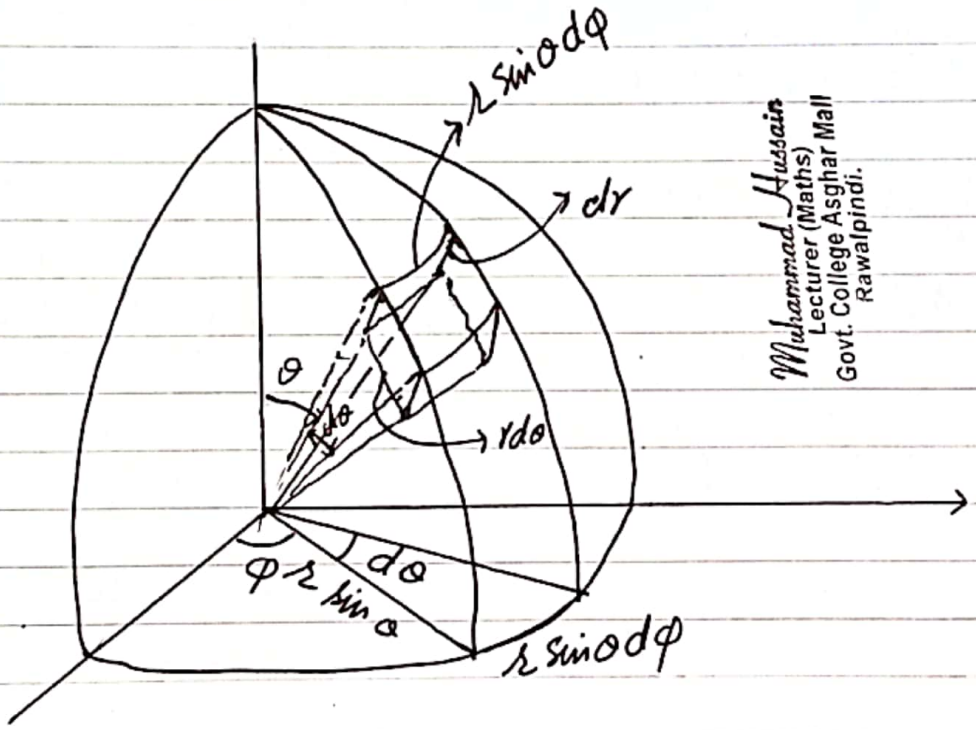

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Expression For Jacobian

We know that Jacobian in an orthogonal curvilinear Co-ordinates u, v, w is given by

$$J\left(\frac{x, y, z}{u, v, w}\right) = h_u h_v h_w$$

In spherical coordinates $h_u = h_r = 1$ $h_v = h_\theta = r$

$h_w = h_\varphi = r \sin\theta$

$$J = (1)(r)(r\sin\theta) = r^2 \sin\theta$$

(h)# Expressions for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in Spherical Coordinates

We know that $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

and $r = \sqrt{x^2+y^2+z^2}$ $\quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$ $\quad \varphi = \tan^{-1}(y/x)$

Then $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\cos\varphi}{r} = \sin\theta\cos\varphi$

$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\sin\varphi}{r} = \sin\theta\sin\varphi$

$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{r\cos\theta}{r} = \cos\theta$

$\frac{\partial\theta}{\partial x} = \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = \frac{(r\sin\theta\cos\varphi)(r\cos\theta)}{r^2(r\sin\theta)}$

$= \frac{\cos\theta\cos\varphi}{r}$

$\frac{\partial\theta}{\partial y} = \frac{yz}{(y^2+x^2+z^2)\sqrt{x^2+y^2}} = \frac{(r\sin\theta\sin\varphi)(r\cos\theta)}{r^2(r\sin\theta)}$

$= \frac{\cos\theta\sin\varphi}{r}$

$\frac{\partial\theta}{\partial z} = \frac{\sqrt{x^2+y^2}}{(\sqrt{y^2+z^2+x^2})^2} = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$

$\frac{\partial\varphi}{\partial x} = -\frac{y}{x^2+y^2} = \frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta} = -\frac{\sin\varphi}{r\sin\theta}$

$\frac{\partial\varphi}{\partial y} = \frac{x}{x^2+y^2} = \frac{r\sin\theta\cos\varphi}{r^2\sin^2\theta} = \frac{\cos\varphi}{r\sin\theta}$

$\frac{\partial\varphi}{\partial z} = 0$

Now by chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x}$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial \theta} + \left(-\frac{\sin\varphi}{r\sin\theta}\right)\frac{\partial}{\partial \varphi} \longrightarrow (1)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y}$$

$$= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + (0)\frac{\partial}{\partial \varphi}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

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(j) Expressions for Gradient, Divergence, Curl, and Laplacian in Spherical Polar Coordinates

We know that for spherical polar coordinates

$u = r \quad v = \theta \quad w = \varphi \quad \hat{e}_u = \hat{e}_r \quad \hat{e}_v = \hat{e}_\theta \quad \hat{e}_w = \hat{e}_\varphi$

$h_u = h_r = 1 \quad h_v = h_\theta = r \quad h_w = h_\varphi = r\sin\theta$

Expression for Gradient

In orthogonal curvilinear coordinates, we have

$$\nabla\psi = h_u\frac{\partial\psi}{\partial u}\hat{e}_u + h_v\frac{\partial\psi}{\partial v}\hat{e}_v + h_w\frac{\partial\psi}{\partial w}\hat{e}_w$$

In spherical polar coordinates, this becomes

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi$$

$$= \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi$$

Expression for Divergence

In orthogonal curvilinear coordinates, we have

$$\nabla \cdot \underline{A} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w A_u) + \frac{\partial}{\partial v}(h_u h_w A_v) + \frac{\partial}{\partial w}(h_u h_v A_w)\right]$$

In spherical polar co-ordinates, this is

$$\nabla \cdot \underline{A} = \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}(r^2 \sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \varphi}(r A_\varphi)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

Expression for Curl

In orthogonal curvilinear Coordinates, we have

$$\nabla \times \underline{A} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u A_u & h_v A_v & h_w A_w \end{vmatrix}.$$

In spherical polar Coordinates, we have

$$\nabla \times \underline{A} = \frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & r A_\varphi & r\sin\theta A_\varphi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial \theta}(r\sin\theta A_\varphi) - \frac{\partial}{\partial \varphi}(r A_\varphi)\right\}\hat{e}_r + \left\{\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(r\sin\theta A_\varphi)\right\} r\hat{e}_\theta \right.$$

$$\left. + \left\{\frac{\partial}{\partial r}(r A_\varphi) - \frac{\partial A_r}{\partial \theta}\right\} r\sin\theta\,\hat{e}_\varphi\right]$$

Expression for Laplacian

In orthogonal curvilinear coordinates, we have

$$\nabla^2\psi = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial\psi}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial\psi}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\psi}{\partial w}\right)\right]$$

In spherical polar coordinates, we have

$$\nabla^2\psi = \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{r}{r\sin\theta}\cdot\frac{\partial\psi}{\partial\varphi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

OR

We know that

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$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi \rightarrow (1)$$

and

$$\nabla\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \rightarrow (2)$$

By (1) and (2)

$$\nabla^2\psi = \nabla\cdot\nabla\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

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Relationship Between Unit Orthogonal Vectors of Circular and Spherical Polar Co-ordinates

We explore a relationship between $\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$ of the circular cylinderical coordinates and $\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$ of the spherical polar coordinates system

We know that in spherical polar co-ordinates

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k} \longrightarrow (1)$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k} \longrightarrow (2)$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} + 0\,\hat{k} \longrightarrow (3)$$

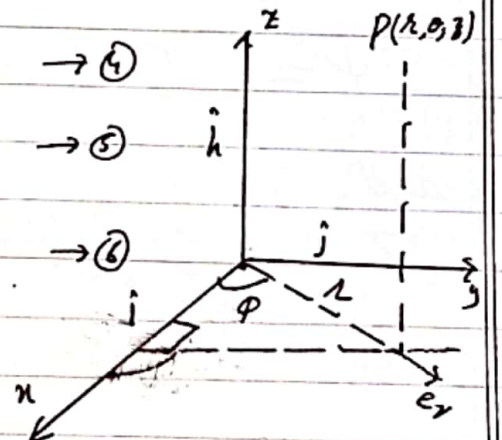
Now $\hat{i}, \hat{j}, \hat{k}$ in cylinderical polar coordinates are

$$\hat{i} = (\hat{i}\cdot\hat{e}_\rho)\,\hat{e}_\rho + (\hat{i}\cdot\hat{e}_\varphi)\,\hat{e}_\varphi + (\hat{i}\cdot\hat{e}_z)\,\hat{e}_z \longrightarrow (4)$$

$$= \cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi + 0\,\hat{e}_z \longrightarrow (5)$$

$$\hat{j} = \sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi \quad , \hat{k} = \hat{k} \longrightarrow (6)$$

using these values in (1), (2) & (3)



$$\hat{e}_\rho = \sin\theta\cos\varphi(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi) + \sin\theta\sin\varphi(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi) + \cos\theta\,\hat{k}$$

$$\hat{e}_\rho = \sin\theta(\cos^2\varphi + \sin^2\varphi)\,\hat{e}_\rho + \cos\theta\,\hat{k}$$

$$\hat{e}_\rho = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{k} \longrightarrow (7)$$

$$\hat{e}_\theta = \cos\theta\cos\varphi(\cos\varphi\,\hat{e}_r - \sin\varphi\,\hat{e}_\varphi) + \cos\theta\sin\varphi(\sin\varphi\,\hat{e}_r + \cos\varphi\,\hat{e}_\varphi) - \sin\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta(\sin^2\varphi+\cos^2\varphi)\hat{e}_r - \sin\theta\hat{k}$$

$$\hat{e}_\theta = \cos\theta\hat{e}_r - \sin\theta\hat{k} \longrightarrow (8)$$

$$\hat{e}_\varphi = -\sin\varphi(\cos\varphi\hat{e}_r - \sin\varphi\hat{e}_\varphi) + \cos\varphi(\sin\varphi\hat{e}_r + \cos\varphi\hat{e}_\varphi)$$

$$\hat{e}_\varphi = (\sin^2\varphi+\cos^2\varphi)\hat{e}_\varphi = \hat{e}_\varphi \longrightarrow (9)$$

In matrix form (7), (8) & (9) can be written as

$$\begin{bmatrix} \hat{e}_\rho \\ \hat{e}_\theta \\ \hat{e}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\varphi \\ \hat{e}_\theta \end{bmatrix}$$

Similarly unit vectors in circular cylindrical coordinates are

$$\hat{e}_r = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \longrightarrow (10)$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j} \longrightarrow (11)$$

$$\hat{e}_\theta = \hat{k} \longrightarrow (12)$$

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Now $\hat{i}, \hat{j}, \hat{k}$ in spherical polar coordinates are

$$\hat{i} = \sin\theta\cos\varphi\,\hat{e}_\rho + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi \longrightarrow (13)$$

$$\hat{j} = \sin\theta\sin\varphi\,\hat{e}_\rho + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi \longrightarrow (14)$$

$$\hat{k} = \cos\theta\,\hat{e}_\rho - \sin\theta\,\hat{e}_\theta \longrightarrow (15)$$

using these in (10), (11) & (12), we have

$$\hat{e}_r = \cos\varphi(\sin\theta\cos\varphi\,\hat{e}_\rho + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi) + \sin\varphi(\sin\theta\sin\varphi\,\hat{e}_\rho + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi)$$

$$\hat{e}_r = \sin\theta\,\hat{e}_\rho + \cos\theta\,\hat{e}_\theta \longrightarrow (16)$$

$$\hat{e}_\varphi = \hat{e}_\varphi \longrightarrow (17)$$

$$\hat{e}_z = \cos\theta\,\hat{e}_\rho - \sin\theta\,\hat{e}_\theta \longrightarrow (18)$$

In matrix notation (16), (17) and (18) can be expressed as

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\varphi \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_\rho \\ \hat{e}_\theta \\ \hat{e}_\varphi \end{bmatrix}$$

Note # we note that the matrix of transformation $(\hat{e}_r, \hat{e}_\varphi, \hat{e}_z) \xrightarrow{T} (\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi)$ is

$$M = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix}$$

and the matrix of inverse transformation $(\hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi) \xrightarrow{T^{-1}} (\hat{e}_r, \hat{e}_\varphi, \hat{e}_z)$ is

$$M^t = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

$$M^t M = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ transformation T is an orthogonal transformation which changes an orthogonal basis to an orthogonal basis and vice versa.

Problem# Represent the vectors $\underline{A} = 2y\hat{i} - 3\hat{j} + 3x\hat{k}$ in spherical polar coordinates and determine A_r, A_θ, A_φ

Sol# We know that in spherical polar coordinates

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

and

$$\hat{i} = \sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi$$

$$\hat{j} = \sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi$$

$$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

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Then $\underline{A} = 2y\hat{i} - 3\hat{j} + 3x\hat{k}$

$$= 2r\sin\theta\sin\varphi(\sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi)$$
$$- r\cos\theta(\sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi)$$
$$+ 3r\sin\theta\cos\varphi(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$$

$$= (2r\sin^2\theta\sin\varphi\cos\varphi - r\sin\theta\cos\theta\sin\varphi + 3r\sin\theta\cos\theta\cos\varphi)\hat{e}_r$$
$$+ (2r\sin\theta\cos\theta\sin\varphi\cos\varphi - r\cos^2\theta\sin\varphi - 3r\sin^2\theta\cos\varphi)\hat{e}_\theta$$
$$+ (+r\cos\theta\cos\varphi - 3r\sin\theta\sin^2\varphi)\hat{e}_\varphi$$

Hence

$$A_r = 2r\sin^2\theta\sin\varphi\cos\varphi - r\sin\theta\cos\theta\sin\varphi + 3r\sin\theta\cos\theta\cos\varphi$$

$$A_\theta = 2r\sin\theta\cos\theta\sin\varphi\cos\varphi - r\cos^2\theta\sin\varphi - 3r\sin^2\theta\cos\varphi$$

$$A_\varphi = -2r\sin\theta\sin^2\varphi - r\cos\theta\cos\varphi$$

Problem # Transform $\vec{A} = \frac{1}{r}\hat{e}_r$ into rectangular cartesian coordinates, where $\vec{A}$ is given in cylinderical coordinates.

Sol # We know that in cylinderical coordinates

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

Then $A = \frac{1}{r}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$\because x = r\cos\theta \quad y = r\sin\theta$ and $r^2 = x^2 + y^2$

$$\underline{A} = \frac{x^2}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j} \quad \text{Ans}$$

Problem # In spherical polar Coordinates, $\underline{A} = \frac{1}{r\sin\theta}\hat{e}_\varphi$, Transform $\vec{A}$ into rectangular coordinates

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Sol # We know that in spherical polar coordinates

$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

Then

$$A = \frac{1}{r\sin\theta} = \frac{1}{r\sin\theta}(-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})$$

$$= -\frac{\sin\varphi}{r\sin\theta}\hat{i} + \frac{\cos\varphi}{r\sin\theta}\hat{j}$$

Since $x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

we have $\sin\varphi = \frac{y}{r\sin\theta}$ $\quad \frac{x}{r\sin\theta} = \cos\varphi$

and $x^2 + y^2 = r^2\sin^2\theta$

$$\Rightarrow \vec{A} = -\frac{y}{r^2\sin^2\theta}\hat{i} + \frac{x}{r^2\sin^2\theta}\hat{j} = -\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\hat{j}$$

Problem# Prove that in cylinderical polar Coordinates

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta \qquad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r$$ where dots denote differentiation w.r.t time t

Sol# We know that in cylinderical polar Coordinates

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\frac{d}{dt}\hat{e}_r = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}$$

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$$= (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt} = \dot{\theta}\,e_\theta$$

and

$$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j} = -(\cos\theta\hat{i} + \sin\theta\hat{j})\frac{d\theta}{dt}$$

$$= -\dot{\theta}\hat{e}_r$$

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Problem# Prove that in spherical Polar Coordinates

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\phi}\,\hat{e}_\varphi$$

$$\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r + \cos\theta\,\dot{\phi}\,\hat{e}_\varphi$$

$$\frac{d}{dt}\hat{e}_\varphi = -\sin\theta\,\dot{\phi}\,\hat{e}_r - \cos\theta\,\dot{\phi}\,\hat{e}_\varphi$$

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Sol# We know that

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

Then $\frac{d}{dt}\hat{e}_r = -\sin\theta\sin\varphi\dot{\varphi}\,\hat{i} + \cos\theta\dot{\theta}\cos\varphi\,\hat{i}$
$+ \sin\theta\cos\varphi\dot{\varphi}\,\hat{j} + \cos\theta\sin\varphi\dot{\theta}\,\hat{j} - \sin\theta\dot{\theta}\,\hat{k}$

$= \dot{\theta}(\cos\theta\cos\varphi\hat{i} + \cos\theta\sin\varphi\hat{j} - \sin\theta\hat{k}) + \sin\theta\dot{\varphi}(-\sin\varphi\hat{i} + \cos\varphi\hat{j})$

$= \dot{\theta}\hat{e}_\theta + \sin\theta\dot{\varphi}\hat{e}_\varphi$

$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\sin\varphi\dot{\varphi}\hat{i} - \sin\theta\dot{\theta}\cos\varphi\hat{i} + \cos\theta\cos\varphi\dot{\varphi}\hat{j}$
$- \sin\theta\dot{\theta}\sin\varphi\hat{j} - \cos\theta\dot{\theta}\hat{k}$

$= -\dot{\theta}(\sin\theta\cos\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\theta\hat{k}) + \cos\theta\dot{\varphi}(-\sin\varphi\hat{i} + \cos\varphi\hat{j})$

$= -\dot{\theta}\hat{e}_r + \cos\theta\dot{\varphi}\hat{e}_\varphi$

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$\frac{d}{dt}\hat{e}_\varphi = -\cos\varphi\dot{\varphi}\hat{i} - \sin\varphi\dot{\varphi}\hat{j}$

Now
$\hat{i} = \sin\theta\cos\varphi\hat{e}_r + \cos\theta\cos\varphi\hat{e}_\theta - \sin\varphi\hat{e}_\varphi$

$\hat{j} = \sin\theta\sin\varphi\hat{e}_r + \cos\theta\sin\varphi\hat{e}_\theta + \cos\varphi\hat{e}_\varphi$

$\frac{d}{dt}\hat{e}_\varphi = -\cos\varphi\dot{\varphi}(\sin\theta\cos\varphi\hat{e}_r + \cos\theta\cos\varphi\hat{e}_\theta - \sin\varphi\hat{e}_\varphi)$
$- \sin\varphi\dot{\varphi}(\sin\theta\sin\varphi\hat{e}_r + \cos\theta\sin\varphi\hat{e}_\theta + \cos\varphi\hat{e}_\varphi)$

$= -\sin\theta\dot{\varphi}\hat{e}_r - \cos\theta\dot{\varphi}\hat{e}_\theta$

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Problem# Express the velocity $\vec{v}$ and acceleration $\vec{a}$ of of a particle in cylinderical polar coordinates.

Sol# In rectangular cartesian coordinates, the pv is

$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and velocity & (vector) acceleration are

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\underline{r} = x\hat{i} + y\hat{i} + z\hat{k}$$

In cylindrical polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

$$\hat{k} = \hat{e}_z$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

$$\underline{r} = r\cos\theta\,(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) + r\sin\theta\,(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) + z\,e_z$$

$$\underline{r} = r\hat{e}_r + z\hat{e}_z$$

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$$\vec{V} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$

~~Now $\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\varphi}\hat{e}_\varphi$ $\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r + \cos\theta\,\dot{\varphi}\hat{e}_\varphi$~~

~~$\vec{V} = \dot{r}\hat{e}_r + r(\dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\varphi}\hat{e}_\varphi)$~~

Now in cylindrical polar coordinates

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta \qquad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r \qquad \frac{d\hat{e}_z}{dt} = \frac{d}{dt}\hat{k} = 0$$

Therefore

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z + 0$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$$

Differentiating again

$$\vec{a} = \frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z\right)$$

$$= \dot{r}\frac{d\hat{e}_r}{dt} + \ddot{r}\hat{e}_r + r\dot{\theta}\frac{d\hat{e}_\theta}{dt} + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\hat{e}_\theta + \ddot{z}\hat{e}_z$$

putting $\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta \qquad \frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\hat{e}_r$

$$\underline{a} = \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{r}\hat{e}_r + r\dot{\theta}(-\dot{\theta})\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{z}e_z + r\ddot{\theta}e_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta + \ddot{z}e_z$$

Problem# Express the velocity $\vec{V}$ and $\vec{a}$ of a particle in spherical polar coordinates:

Sol# In rectangular cartesian coordinates, the position vector is

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and the velocity and acceleration vectors are

$$\vec{V} = \frac{d\underline{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

and $\underline{a} = \frac{d^2\underline{r}}{dt^2} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$

In spherical polar Co-ordinates

$$\vec{r} = r\hat{e}_r$$

$$\underline{V} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

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In spherical polar coordinates

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\varphi}\hat{e}_\varphi$$

$$\underline{V} = \dot{r}\hat{e}_r + r(\dot{\theta}\hat{e}_\theta + \sin\theta\,\dot{\varphi}\hat{e}_\varphi)$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\varphi}\hat{e}_\varphi$$

Differentiating again.

$$\underline{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + r\dot{\theta}\frac{d\hat{e}_\theta}{dt} + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta +$$

$$+ r\sin\theta\ddot{\phi}\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi + \dot{r}\sin\theta\hat{e}_\phi + r\sin\theta\frac{d\hat{e}_\phi}{dt}\cdot\dot{\phi}$$

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putting $\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta + \sin\theta\dot{\phi}\hat{e}_\phi$

$$\frac{d\hat{e}_\phi}{dt} = -\sin\theta\dot{\phi}\hat{e}_r - \cos\theta\dot{\phi}\hat{e}_\theta$$

$$= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \sin\theta\dot{\phi}\hat{e}_\phi) + r\dot{\theta}(-\dot{\theta}\hat{e}_r + \cos\theta\dot{\phi}\hat{e}_\phi)$$

$$+ r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\sin\theta(-\sin\theta\dot{\phi}\hat{e}_r - \cos\theta\dot{\phi}\hat{e}_\theta)\dot{\phi}$$
$$+ r\sin\theta\ddot{\phi}\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi + \dot{r}\sin\theta\dot{\phi}\hat{e}_\phi$$

$$= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{e}_\theta$$
$$+ (2\dot{r}\sin\theta\dot{\phi} + 2r\dot{\theta}\cos\theta\dot{\phi} + r\sin\theta\ddot{\phi})\hat{e}_\phi$$

$$= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{e}_r + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2\right]\hat{e}_\theta$$

$$+ \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi})\hat{e}_\phi$$

Problem # In cylinderical Polar Coordinates, show that

$$\frac{\partial\hat{e}_r}{\partial r} = 0 \quad \frac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta \quad \frac{\partial\hat{e}_z}{\partial z} = 0$$

$$\frac{\partial\hat{e}_\theta}{\partial r} = 0 \quad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r \quad \frac{\partial\hat{e}_\theta}{\partial z} = 0$$

$$\frac{\partial\hat{e}_z}{\partial r} = 0 \quad \frac{\partial\hat{e}_z}{\partial\theta} = 0 \quad \frac{\partial\hat{e}_z}{\partial z} = 0$$

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Sol # We know that

$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$ $\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$ and $\hat{e}_z = \hat{k}$

Then $\frac{\partial\hat{e}_r}{\partial r} = 0$ $\frac{\partial\hat{e}_\theta}{\partial\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} = \hat{e}_\theta$
$= -\sin\theta\hat{i} + \cos\theta\hat{j} = \hat{e}_\theta$

$\frac{\partial \hat{e}_r}{\partial \theta} = 0$ $\quad \frac{\partial \hat{e}_\theta}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\theta \hat{i} - \sin\theta \hat{j} = -\hat{e}_r$

$\frac{\partial \hat{e}_\theta}{\partial \theta} = 0$ $\quad \frac{\partial \hat{e}_z}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_z}{\partial \theta} = 0$

Problem# In spherical Polar coordinates, show that

$\frac{\partial \hat{e}_r}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$ $\quad \frac{\partial \hat{e}_r}{\partial \varphi} = \sin\theta \hat{e}_\varphi$

$\frac{\partial \hat{e}_\theta}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$ $\quad \frac{\partial \hat{e}_\theta}{\partial \varphi} = \cos\theta \hat{e}_\varphi$

$\frac{\partial \hat{e}_\varphi}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_\varphi}{\partial \theta} = 0$ $\quad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\sin\theta \hat{e}_r - \cos\theta \hat{e}_\theta$

Sol# We know that

$$\hat{e}_r = \sin\theta\cos\varphi \hat{i} + \sin\theta\sin\varphi \hat{j} + \cos\theta \hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\varphi \hat{i} + \cos\theta\sin\varphi \hat{j} - \sin\theta \hat{k}$$

$$\hat{e}_\varphi = -\sin\varphi \hat{i} + \cos\varphi \hat{j}$$

Then $\frac{\partial \hat{e}_r}{\partial r} = 0$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \cos\theta\cos\varphi \hat{i} + \cos\theta\sin\varphi \hat{j} - \sin\theta \hat{k} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \varphi} = -\sin\theta\sin\varphi \hat{i} + \sin\theta\cos\varphi \hat{j} = \sin\theta(-\sin\varphi \hat{i} + \cos\varphi \hat{j}) = \sin\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\sin\theta\cos\varphi \hat{i} - \sin\theta\sin\varphi \hat{j} - \cos\theta \hat{k} = -(\sin\theta\cos\varphi \hat{i} + \sin\theta\sin\varphi \hat{j} + \cos\theta \hat{k}) = -\hat{e}_r$$

$$\frac{\partial \hat{e}_\theta}{\partial \varphi} = -\cos\theta\sin\varphi \hat{i} + \cos\theta\cos\varphi \hat{j} - (\cancel{\cos\theta \hat{k}})^{0} = \cos\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial r} = 0 \qquad \frac{\partial \hat{e}_\varphi}{\partial \theta} = 0$$

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$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\varphi\, \hat{i} - \sin\varphi\, \hat{j}$$

putting $\hat{i} = \sin\theta\cos\varphi\, \hat{e}_r + \cos\theta\cos\varphi\, \hat{e}_\theta - \sin\varphi\, \hat{e}_z$

$\hat{j} = \sin\theta\sin\varphi\, \hat{e}_r + \cos\theta\sin\varphi\, \hat{e}_\theta + \cos\varphi\, \hat{e}_z$

$\hat{k} =$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\varphi\,(\sin\theta\cos\varphi\, \hat{e}_r + \cos\theta\cos\varphi\, \hat{e}_\theta - \sin\varphi\, \hat{e}_z)$$
$$-\sin\varphi\,(\sin\theta\sin\varphi\, \hat{e}_r + \cos\theta\sin\varphi\, \hat{e}_\theta + \cos\varphi\, \hat{e}_z)$$

$$= -\sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta$$

Problem# Find ∇r $\nabla\theta$ $\nabla\varphi$ in spherical coordinate system and show that

$\hat{E}_r = \hat{e}_r$ $\hat{E}_\theta = \hat{e}_\theta$ $\hat{E}_\varphi = \hat{e}_\varphi$ For this system

Sol# We know that in spherical coordinate system

$u = r$ $v = \theta$ $\omega = \varphi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{e}_\varphi$$

for $\psi = r$

$$\nabla r = \frac{\partial r}{\partial r}\hat{e}_r + 0 + 0 = \hat{e}_r$$

$\because r, \theta, \varphi$ are independent variables

Similarly for $\psi = \theta$

$$\nabla\theta = \frac{1}{r}\hat{e}_\theta \quad \& \quad \nabla\varphi = \frac{1}{r\sin\theta}\hat{e}_\varphi$$

Then $|\nabla r| = |\hat{e}_r| = 1 = \frac{1}{h_r}\frac{1}{1}$

$|\nabla\theta| = |\frac{1}{r}\hat{e}_\theta| = \frac{1}{r} = \frac{1}{h_\theta}$

$|\nabla\phi| = |\frac{1}{r\sin\theta}\hat{e}_\phi| = \frac{1}{r\sin\theta} = \frac{1}{h_\phi}$

$\hat{E}_r = \frac{\nabla r}{|\nabla r|} = \frac{\hat{e}_r}{1}$

$\hat{E}_\theta = \frac{\nabla\theta}{|\nabla\theta|} = \frac{1}{r}\hat{e}_\theta \cdot r = \hat{e}_\theta$

$\hat{E}_\phi = \frac{\nabla\phi}{|\nabla\phi|} = r\sin\theta\frac{1}{r\sin\theta}\hat{e}_\phi = \hat{e}_\phi$

Problem # Find ∇r, $\nabla\theta$, ∇z in cylinderical coordinate system and show that

$\hat{E}_r = \hat{e}_r \quad \hat{E}_\theta = \hat{e}_\theta \quad \hat{E}_z = \hat{e}_z$

Sol # We know that in cylinderical coordinate system

$u = r \quad v = \theta \quad w = z$

and

$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z$

for $\psi = r$

$\nabla r = \hat{e}_r$

for $\psi = \theta \quad \nabla\theta = \frac{1}{r}\hat{e}_\theta$

$\nabla z = \hat{e}_z$

$|\nabla r| = 1 \quad |\nabla\theta| = \frac{1}{r} \quad |\nabla z| = 1$

$\hat{E}_r = \frac{\nabla r}{|\nabla r|} = \frac{\hat{e}_r}{1} = \hat{e}_r$

$\hat{E}_\theta = \frac{\nabla\theta}{|\nabla\theta|} = r\cdot\frac{1}{r}\hat{e}_\theta = \hat{e}_\theta$

$E_z = \frac{\nabla z}{|\nabla z|} = \frac{\hat{e}_z}{1} = \hat{e}_z$

Problem # If $\vec{A} = r^2\cos\theta\,\hat{e}_r + \frac{1}{r}\hat{e}_\theta + \frac{1}{r\sin\theta}\hat{e}_\varphi$

Find (i) $\nabla\cdot\underline{A}$ (ii) $\nabla\times\underline{A}$ in spherical polar coordinates.

Sol # Given that $\vec{A} = r^2\cos\theta\,\hat{e}_r + \frac{1}{r}\hat{e}_\theta + \frac{1}{r\sin\theta}\hat{e}_\varphi$

$\Rightarrow A_r = r^2\cos\theta \quad A_\theta = \frac{1}{r} \quad A_\varphi = \frac{1}{r\sin\theta}$

In spherical polar coordinates we have

$$\nabla\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r\sin\theta}\right)$$

$$= \frac{1}{r^2}(4r^3\cos\theta) + \frac{1}{r^2\sin\theta}\cdot\cos\theta + 0$$

$$= 4r\cos\theta + \frac{1}{r^2}\cot\theta$$

$$\nabla\times\underline{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta\,A_\varphi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ r^2\cos\theta & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[0 + 0 + r\sin\theta(0 + r^2\sin\theta)\hat{e}_\varphi\right]$$

$$= \frac{1}{r^2\sin\theta}\left[r^3\sin^2\theta\right]\hat{e}_\varphi = r\sin\theta\,\hat{e}_\varphi \quad \text{Ans}$$

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Problem# Show that in spherical polar coordinates, $\nabla \times (\cos\theta \nabla\phi) = \nabla(\frac{1}{r})$

Sol We know that in spherical polar coordinates

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi \longrightarrow (1)$$

Let $\psi = \phi$ $\quad \nabla\phi = \frac{1}{r\sin\theta}\hat{e}_\phi$

$$\cos\theta\nabla\phi = \frac{\cos\theta}{r\sin\theta}\hat{e}_\phi = \frac{1}{r}\cot\theta\,\hat{e}_\phi$$

$$\nabla\times(\cos\theta\nabla\phi) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \cos\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}(-\sin\theta\,\hat{e}_r) = -\frac{1}{r^2}\hat{e}_r$$

taking $\psi = \frac{1}{r}$ in (1)

$$\nabla(\frac{1}{r}) = \frac{\partial}{\partial r}(\frac{1}{r})\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{1}{r})\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\frac{1}{r})\hat{e}_\phi$$

$$= -\frac{1}{r^2}\hat{e}_r + 0 + 0 = -\frac{1}{r^2}\hat{e}_r$$

Problem# In cylinderical polar coordinates, a vector field $\underline{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_z\hat{e}_z$ where the components A_r, A_θ, A_z are independent of z show that if $\hat{e}_z \cdot \nabla\times(\nabla\times\underline{A}) = 0$, then $r\frac{\partial^2 A_z}{\partial r^2} + \frac{\partial A_z}{\partial r} + \frac{1}{r}\frac{\partial^2 A_z}{\partial\theta^2} = 0$

Sol# We have

$$\nabla \times \underline{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\left[\frac{\partial A_z}{\partial \theta}\hat{e}_r - r\frac{\partial A_z}{\partial z}\hat{e}_\theta + r\hat{e}_\theta(0-0) + \hat{e}_z\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{r\partial \theta}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial A_z}{\partial \theta}\hat{e}_r - 0 - r\frac{\partial A_z}{\partial z}\hat{e}_\theta + 0 + \left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right\}\hat{e}_z\right]$$

$$\nabla \times (\nabla \times \underline{A}) = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{r}\frac{\partial A_z}{\partial \theta} & -r\frac{\partial A_z}{\partial z} & \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{vmatrix}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-r\frac{\partial A_z}{\partial r}\right) - \frac{\partial}{\partial \theta}\frac{1}{r}\left(\frac{\partial A_z}{\partial \theta}\right)\right]$$

Forom $\hat{e}_z \cdot \nabla \times (\nabla \times \underline{A}) = 0$, we obtain

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(-r\frac{\partial A_z}{\partial r}\right) - \frac{\partial}{\partial \theta}\frac{1}{r}\left(\frac{\partial A_z}{\partial \theta}\right)\right] = 0$$

$$\frac{1}{r}\left[-r\frac{\partial^2 A_z}{\partial r^2} - \frac{\partial A_z}{\partial r} - \frac{1}{r}\frac{\partial^2 A_z}{\partial \theta^2}\right] = 0$$

$$\frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} = 0$$

$$\Rightarrow r\frac{\partial^2 A_z}{\partial r^2} + \frac{\partial A_z}{\partial r} + \frac{1}{r}\frac{\partial^2 A_z}{\partial \theta^2} = 0$$

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Problem # If $\psi(r,z)$ is a function of r and z only, in cylinderical polar coordinates r,θ,z, show that the vector field $\vec{A} = -\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial r}\hat{e}_z$ satisfies equation $\nabla\cdot\underline{A} = 0$

and $\nabla\times\underline{A} = \left(\frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\right)\hat{e}_\theta$

Sol # Given that

$$\vec{A} = -\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial r}\hat{e}_z$$

$$\Rightarrow A_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad A_\theta = 0 \qquad A_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$$

In cylinderical polar coordinates

$$\nabla\cdot\underline{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{\partial\theta} + \frac{\partial}{\partial z}(rA_z)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{\partial\psi}{\partial z}\right) + 0 + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)\right]$$

$$= \frac{1}{r}\left[-\frac{\partial^2\psi}{\partial r\partial z} + \frac{\partial^2\psi}{\partial z\partial r}\right] = 0$$

provided the order of differentiation is immaterial

$$\nabla\times\underline{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ -\frac{1}{r}\frac{\partial\psi}{\partial z} & 0 & \frac{1}{r}\frac{\partial\psi}{\partial r} \end{vmatrix}$$

$$= \frac{1}{r}\left[(0)\hat{e}_r + \left\{-\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\right\}r\hat{e}_\theta + (0)\hat{e}_z\right]$$

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$$= \left(\frac{1}{r^2}\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2}\right)\hat{e}_\theta$$

Problem# A vector field $\vec{A}$ is such that in terms of cylinderical polar coordinates r, θ, z

$\underline{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta$, where A_r, A_θ are independent of z. Express $(\underline{A}\cdot\nabla)\underline{A}$, $\frac{1}{2}\nabla A^2$ and $\underline{A}\times(\nabla\times\underline{A})$ in terms of cylinderical components and varify that identity

$$(\underline{A}\cdot\nabla)\underline{A} = \frac{1}{2}\nabla^2 A^2 - \underline{A}\times(\nabla\times\underline{A})$$ is satisfied.

Sol# Given that $\underline{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta$

In cylinderical polar coordinates, we have

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Then

$$(\vec{A}\cdot\nabla)\underline{A} = \left[(A_r\hat{e}_r + A_\theta\hat{e}_\theta)\cdot\left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\right]\underline{A}$$

$$= \left(A_r\frac{\partial}{\partial r} + A_\theta\cdot\frac{1}{r}\frac{\partial}{\partial \theta} + 0\right)(A_r\hat{e}_r + A_\theta\hat{e}_\theta)$$

$$= A_r\frac{\partial A_r}{\partial r}\hat{e}_r + A_r^2\frac{\partial \hat{e}_r}{\partial r} + A_r\frac{\partial A_\theta}{\partial r}\hat{e}_\theta + A_rA_\theta\frac{\partial \hat{e}_\theta}{\partial r}$$

$$+ A_\theta\frac{1}{r}\frac{\partial A_r}{\partial \theta}\hat{e}_r + \frac{A_\theta A_r}{r}\frac{\partial \hat{e}_r}{\partial \theta} + \frac{A_\theta}{r}\frac{\partial A_\theta}{\partial \theta}\hat{e}_\theta + \frac{A_\theta^2}{r}\frac{\partial \hat{e}_\theta}{\partial r}$$

But $\frac{\partial \hat{e}_r}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_\theta}{\partial r} = 0$ $\quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$

$$\frac{\partial \hat{e}_\theta}{\partial r} = -\hat{e}_r$$

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so

$$(\vec{A}\cdot\nabla)\underline{A} = A_r\frac{\partial A_r}{\partial r}\hat{e}_r + A_r\frac{\partial A_\theta}{\partial r}\hat{e}_\theta + \frac{A_\theta}{r}\frac{\partial A_r}{\partial \theta}\hat{e}_r + A_\theta A_r\hat{e}_\theta$$

$$+ A_\theta\cdot\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}\hat{e}_\theta - \frac{A_\theta^2}{r}\hat{e}_r$$

$$= \left(A_r\frac{\partial A_r}{\partial r} + A_\theta\cdot\frac{1}{r}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta^2}{r}\right)\hat{e}_r + \left(A_r\frac{\partial A_\theta}{\partial r} + \frac{1}{r}A_rA_\theta + A_\theta\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}\right)\hat{e}_\theta \quad \rightarrow (1)$$

Now $A^2 = \underline{A}\cdot\underline{A} = (A_s \hat{e}_s + A_\phi \hat{e}_\phi)\cdot(A_s \hat{e}_s + A_\phi \hat{e}_\phi)$

$= A_s^2 + A_\phi^2$

So $\frac{1}{2}\nabla A^2 = \frac{1}{2}\left(\hat{e}_s \frac{\partial}{\partial s} + \hat{e}_\phi \frac{1}{s}\frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z}\right)(A_s^2 + A_\phi^2)$

$= \frac{1}{2}\left(2A_s \frac{\partial A_s}{\partial s}\hat{e}_s + 2A_\phi \frac{\partial A_\phi}{\partial s}\hat{e}_s + 2A_s \cdot \frac{1}{s}\frac{\partial A_s}{\partial \phi}\hat{e}_\phi + \frac{2A_\phi}{s}\frac{\partial A_\phi}{\partial \phi}\hat{e}_\phi + 0 + 0\right)$

Since A_s & A_ϕ are independent of z

$= \left(A_s \frac{\partial A_s}{\partial s} + A_\phi \frac{\partial A_\phi}{\partial s}\right)\hat{e}_s + \left(A_s \frac{1}{s}\frac{\partial A_s}{\partial \phi} + A_\phi \cdot \frac{1}{s}\frac{\partial A_\phi}{\partial \phi}\right)\hat{e}_\phi$ ⟶ (2)

Next

$$\nabla \times \underline{A} = \frac{1}{s}\begin{vmatrix} \hat{e}_s & s\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & sA_\phi & 0 \end{vmatrix}$$

$= \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{e}_z = \left(\frac{\partial A_\phi}{\partial s} + \frac{1}{s}A_\phi - \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right)e_z$

Therefore

$\underline{A} \times (\nabla \times \underline{A}) = (A_s \hat{e}_s + A_\phi \hat{e}_\phi) \times \left(\frac{\partial A_\phi}{\partial s} + \frac{1}{s}A_\phi - \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right)\hat{e}_z$

$= \left(A_s \frac{\partial A_\phi}{\partial s} + \frac{1}{s}A_s A_\phi - A_\phi \cdot \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right)(\hat{e}_z \times \hat{e}_z)$

$+ \left(A_\phi \frac{\partial A_\phi}{\partial s} + \frac{1}{s}A_\phi^2 - A_\phi \cdot \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right)(\hat{e}_\phi \times \hat{e}_z)$

$= \left(A_\phi \frac{\partial A_\phi}{\partial s} + \frac{1}{s}A_\phi^2 - A_\phi \cdot \frac{1}{s}\frac{\partial A_s}{\partial \phi}\right)\hat{e}_s + \left(\frac{A_s}{s}\frac{\partial A_r}{\partial \phi} - \frac{1}{s}A_r A_\phi - A_r \frac{\partial A_\phi}{\partial s}\right)\hat{e}_\phi$ ⟶ (3)

$\because \hat{e}_s \times \hat{e}_z = -\hat{e}_\phi$ & $\hat{e}_\phi \times \hat{e}_z = \hat{e}_s$

(3) − (2) ⇒

$\frac{1}{2}\nabla A^2 - \underline{A} \times (\nabla \times \underline{A}) = \left(A_r \frac{\partial A_r}{\partial r} + A_\phi \frac{1}{r}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}A_\phi^2\right)\hat{e}_s + \left(A_r \frac{\partial A_\phi}{\partial s} + \frac{A_r A_\phi}{r} +\right.$

$+ A_\theta \frac{1}{r}\frac{\partial A_\theta}{\partial \theta})\hat{e}_\theta \longrightarrow (4)$

Comparing (1) & (4) we note that

$$(\underline{A}\cdot\nabla)\underline{A} = \frac{1}{2}\nabla A^2 - \underline{A}\times(\nabla\times\underline{A})$$

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Parabolic Cylinderical Coordinates

Problem# Parabolic cylinderical Coordinates (u,v,z) are related to cartesian co-ordinates as

$$x = \frac{u^2-v^2}{2} \qquad y = uv \qquad z = z$$

$$-\infty < u < \infty \qquad v \geq 0 \qquad -\infty < z < \infty$$

Calculate the scale factors h_u, h_v, h_z for these. Also find the expression for ∇ in terms of (u, v, z)

Sol# $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= \left(\frac{u^2-v^2}{2}\right)\hat{i} + uv\hat{j} + z\hat{k}$$

$\frac{\partial \underline{r}}{\partial u} = u\hat{i} + v\hat{j}$ $\quad h_u = \left|\frac{\partial \underline{r}}{\partial u}\right| = \sqrt{u^2+v^2}$

$\frac{\partial \underline{r}}{\partial v} = -v\hat{i} + u\hat{j}$ $\quad h_v = \left|\frac{\partial \underline{r}}{\partial v}\right| = \sqrt{u^2+v^2}$

$\frac{\partial \underline{r}}{\partial z} = \hat{k}$ $\quad h_z = \left|\frac{\partial \underline{r}}{\partial z}\right| = 1$

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we have

$$\nabla\psi = \frac{1}{h_u}\frac{\partial \psi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial \psi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial \psi}{\partial w}\hat{e}_w$$

$$\nabla = \frac{1}{h_u}\frac{\partial}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial}{\partial v}\hat{e}_v + \frac{1}{h_z}\frac{\partial}{\partial z}\hat{e}_z$$

$$= \frac{\hat{e}_u}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u} + \frac{\hat{e}_v}{\sqrt{u^2+v^2}}\frac{\partial}{\partial v} + \frac{\hat{e}_z}{1}\frac{\partial}{\partial z}$$

Paraboloidal Coordinates (u, v, ϕ)

$x = uv\cos\phi \qquad y = uv\sin\phi \qquad z = \frac{1}{2}(u^2-v^2)$

where $u \geq 0 \quad v \geq 0 \qquad 0 \leq \phi \leq 2\pi$

$h_u = h_v = \sqrt{u^2+v^2} \qquad h_\phi = uv$

Elliptic Cylinderical Coordinates # (u, v, z)

Problem # Elliptic cylinderical coordinates (u, v, z) are related to cartesian coordinates as

$x = a \cosh u \cos v \qquad y = a \sinh u \sin v \qquad z = z$

Where $u \neq 0 \quad 0 \leq v \leq 2\pi, \quad -\infty < z < \infty$

Calculate the scale factors for (u, v, z)

Sol #

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= a\cosh u \cos v\,\hat{i} + a \sinh u \sin v\,\hat{j} + z\hat{k}$$

$$\frac{\partial \underline{r}}{\partial u} = a \sinh u \cos v\,\hat{i} + a \cosh u \sin v\,\hat{j} + 0\hat{k}$$

$$\left|\frac{\partial \vec{r}}{\partial u}\right| = \sqrt{a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v}$$

$$= \sqrt{a^2 \sinh^2 u (1 - \sin^2 v) + a^2 \cosh^2 u \sin^2 v}$$

$$= \sqrt{a^2 \sinh^2 u - a^2 \sinh^2 u \sin^2 v + a^2 \cosh^2 u \sin^2 v}$$

$$h_u = \sqrt{a^2 \sinh^2 u + a^2 \sin^2 v}$$

$$\frac{\partial \underline{r}}{\partial v} = -a \cosh u \sin v\,\hat{i} + a \sinh u \cos v\,\hat{j} + 0\hat{k}$$

$$\left|\frac{\partial \underline{r}}{\partial v}\right| = \sqrt{a^2 \cosh^2 u \sin^2 v + a^2 \sinh^2 u \cos^2 v}$$

$$= \sqrt{a^2 \cosh^2 u \sin^2 v + a^2 \sinh^2 u (1 - \sin^2 v)}$$

$$= a\sqrt{\sinh^2 v + \sin^2 v}$$

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Bipolar Coordinates # (u, v, z)

Problem # Relation between bipolar and cartesian coordinates is

$x^2+(y-a\cot u)^2 = a^2\operatorname{cosec}^2 u$

$(x-a\coth v)^2+y^2 = a^2\operatorname{cosech}^2 v \qquad z=z$

where $0 \le u \le 2\pi \quad -\infty < v < \infty \quad -\infty < z < \infty$

Calculate the scale factors.

Sol $\underline{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$\frac{\partial \underline{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{i}+\frac{\partial y}{\partial u}\hat{j}+0\hat{k}$

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Now from $x^2+(y-a\cot u)^2 = a^2\operatorname{cosec}^2 u.$

Diff wrt 'u'

$(x-a\coth v)^2+y^2 = a^2\operatorname{cosech}^2 v$

We have

$x = \frac{a\sinh v}{\cosh v-\cos u} \qquad y = \frac{a\sin u}{\cosh v-\cos u} \qquad z=z$

$\frac{\partial x}{\partial u} = a\left[\frac{0-\sinh v(0+\sin u)}{(\cosh v-\cos u)^2}\right]$

$= -\frac{a\sinh v\sin u}{(\cosh v-\cos u)^2}$

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$\frac{\partial y}{\partial u} = a\left[\frac{(\cosh v-\cos u)\cos u - \sin u(0+\sin u)}{(\cosh v-\cos u)^2}\right]$

$= a\left[\frac{\cosh v\cos u-\cos^2 u-\sin^2 u}{(\cosh v-\cos u)^2}\right]$

$= a\left[\frac{\cosh v\cos u-1}{(\cosh v-\cos u)^2}\right]$

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$\left|\frac{\partial \underline{r}}{\partial u}\right| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2}$

$h_u = \sqrt{\frac{a^2\sinh^2 v\sin^2 u + a^2(\cosh^2 v\cos^2 u+1-2\cosh v\cos u)}{(\cosh v-\cos u)^4}}$

$$= \frac{a}{(\cosh v - \cos u)^2}\sqrt{(\cosh^2 v - 1)\sin^2 u + \cosh^2 v \cos^2 u + 1 - 2\cosh v \cos u}$$

$$= \frac{a}{(\cosh v - \cos u)^2}\sqrt{\cosh^2 v - \sin^2 u + 1 - 2\cosh v \cos u}$$

$$= \frac{a}{(\cosh v - \cos u)^2}\sqrt{\cosh^2 v - (1 - \cos^2 u) + 1 - 2\cosh v \cos u}$$

$$= \frac{a}{(\cosh v - \cos u)^2}\sqrt{\cosh^2 v + \cos^2 u - 2\cosh v \cos u}$$

$$= \frac{a}{(\cosh v - \cos u)^2}\sqrt{(\cosh v - \cos u)^2} = \frac{a}{\cosh v - \cos u}$$

Similarly $h_v = \frac{a}{\cosh v - \cos u}$

$h_z = 1$

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Problem Calculate the scale factors h_u, h_v, h_w for the coordinates (u, v, w), where

$x = e^{\sin u \sin v}$ $\quad y = e^{\cos u \cos v}$ $\quad z = z$

Sol $\frac{\partial x}{\partial u} = \cos u \sin v\, e^{\sin u \sin v}$

$\frac{\partial y}{\partial u} = -\sin u \cos v\, e^{\cos u \cos v}$

$\frac{\partial z}{\partial u} = 0$

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$$h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$$

$$= \left(\cos^2 u \sin^2 v\, e^{2\sin u \sin v} + \cos^2 v \sin^2 u\, e^{2\cos u \cos v}\right)^{1/2}$$

Also

$$h_v = \left(\sin^2 u \cos^2 v\, e^{2\sin u \sin v} + \cos^2 u \sin^2 v\, e^{2\cos u \cos v}\right)^{1/2}$$

$h_z = 1$

تمت بالخير #